Interpolation and Extrapolation

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Abstract

We define Lorentz-Zygmund spaces, generalized Lorentz-Zygmund spaces, slowly varying functions, and Lorentz-Karamata spaces. To get an interpolation characterization of Lorentz-Karamata spaces, we examine the K- and the J-method of real interpolation with function parameters in quasi-Banach spaces. In particular, we study the Kalugina class $B_{\mathcal{K}}$ and prove the Equivalence Theorem and the Reiteration Theorem with function parameters.

Finally, we define the Δ and Σ method of extrapolation and achieve an extrapolation characterization that allows us, in particular, to characterize generalized Lorentz-Zygmund spaces by Lorentz spaces.

Contents

Introduction

For $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, and $\Omega \subseteq \mathbb{R}^n$ with finite Lebesgue measure, the *Lorentz-Zygmund space* $L_{p,q}(\log)_\alpha(\Omega)$ is the space of all complex-valued functions f on Ω such that

$$\|f|L_{p,q}(\log L)_\alpha(\Omega)\| = \left(\int_0^\infty \left\{t^{\frac{1}{p}}(1+|\log t|)^\alpha f^*(t)\right\}^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty.$$

Here f^* denotes the non-increasing rearrangement of f.

In [4] an *extrapolation characterization* of these spaces is developed. This extrapolation has been introduced before in [18] in a very general context. It says, for $\alpha < 0$ and $\dfrac{1}{p^{\mu_j}} := \frac{1}{p} + 2^{-j}$, that

$$\left(\sum_{j=j_0}^\infty 2^{j\alpha p} \, \|f|L_{p^{\mu_j},q}(\Omega)\|^p\right)^{1/p}$$

is an equivalent quasi-norm in $L_{p,q}(\log)_\alpha(\Omega)$. A similar method can be used for the case $\alpha > 0$. Such, we can transfer properties of Lorentz spaces to Lorentz-Zygmund spaces.

This is proved by characterizing both the spaces $L_{p^{\mu_j},q}(\Omega)$ and $L_{p,q}(\log)_\alpha(\Omega)$ by interpolation of the same (ordered) couple of spaces $\big(L_\infty(\Omega), L_r(\Omega)\big)$. To this end one has to apply a more general interpolation method than classical real interpolation that works with *function parameters*.

In this thesis, we generalize the above extrapolation characterisation by replacing $(1+|\log t|)^\alpha$ by

$$\prod_{i=1}^N l_i(t)^{\alpha_i},$$

where $l_1(t) = 1 + |\log t|$ and $l_i(t) = 1 + \big|\log\big(l_{i-1}(t)\big)\big|$ for $i = 2, 3, \ldots$.

In [7], the resulting spaces have been called *generalized Lorentz-Zygmund spaces.*

The thesis is organized as follows. In the first chapter we describe some basic facts that we will need. Moreover, we define the so called *Lorentz-Karamata spaces* as the space of all functions f on Ω with

$$\|f|L_{p,q;b}(\Omega)\| = \left(\int_0^\infty \{t^{\frac{1}{p}} b(t) f^*(t)\}^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty.$$

The appearing function b is a *slowly varying function* (See Definition 1.4.1). For example, the above mentioned product of iterated logarithms is slowly varying. So, Lorentz-Karamata spaces are a further generalization of generalized Lorentz-Zygmund spaces.

In the second chapter we treat interpolation with a function parameter. There, we first introduce the Kalugina class $B_{\mathcal{K}}$ of admissible parameter functions and prove some basic properties of this interpolation method such as the Equivalence Theorem, the Reiteration Theorem and the Interpolation Theorem. We follow mainly [13], but consider the case of quasi-Banach spaces and use functions that are merely equivalent to a function in $B_{\mathcal{K}}$. So, we can also use slowly varying functions as parameter functions and characterize Lorentz-Karamata spaces as interpolation spaces.

In the third chapter, following [4] and [18], we examine the Δ and Σ method of extrapolation. Firstly, we extrapolate abstract interpolation spaces. There we generalize the results of [4] to get

$$A_{\theta,\alpha_1,\dots,\alpha_{N-1},\alpha_N,q} \sim \left(\sum_{j=1}^{\infty} 2^{-j\alpha_N q} \|a|A_{\theta,\alpha_1,\dots,\alpha_{N-1}+2^{-j},q}\|^q\right)^{\frac{1}{q}},$$

where $A_{\theta,\alpha_1,\dots,\alpha_{N-1},\alpha_N,q}$ denote the interpolation spaces that use the above mentioned product of iterated logarithms as function parameters. By induction, we get a characterization of $A_{\theta,\alpha_1,\dots,\alpha_{N-1},\alpha_N,q}$ by means of classical real interpolation spaces. Then we apply this theory to concrete function spaces and get an extrapolation characterization of generalized Lorentz-Zygmund spaces based on Lorentz spaces.

1 Preliminaries

1.1 Notations

As usual, we denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the natural, real, and complex numbers.

For two positive functions f and g, defined on $(0,\infty)$, we write $f \lesssim g$ if there is a constant $c > 0$ such that $f(t) \leq cg(t)$ for all $t \in (0,\infty)$. Analogously, we define $f \gtrsim g$. If $f \lesssim g$ and $f \gtrsim g$, we say that f and g are *equivalent* and write $f \sim g$. Sometimes we will write $f(t)$ instead of f even if we refer to the function and not to the value of f at the point t. We will do that for the sake of clarity, especially if f is given as a product or a composition of some functions.

By $\log t$ we denote the logarithm to the basis 2 of t and by $\ln t$ the natural logarithm of t.

For a measurable set $A \subseteq \mathbb{R}^d$ we denote by $|A|$ its d-dimensional Lebesgue measure. For a given set A we choose d as small as possible, for example, if A is an interval of real numbers, $|A|$ denotes the one-dimensional Lebesgue measure.

Mostly, we will denote positive constants by c. This c can be different from line to line.

If an assertion holds for all $0 < q \leq \infty$, we will write down the proof only for $q < \infty$ if the proof for $q = \infty$ follows by the same arguments or is very simple.

1.2 Quasi-Banach spaces

Definition 1.2.1. Let A be a complex vector space. A functional $\|\cdot|A\|\colon A \to [0,\infty)$ is called a *quasi-norm*, if

(i) $\|a|A\| = 0 \quad \text{iff} \quad a = 0$,

(ii) $\|\lambda a|A\| = |\lambda|\|a|A\|$ for all $\lambda \in \mathbb{C}$ and $a \in A$, and

(iii) there is a constant $c \geq 1$ such that $\|a+b|A\| \leq c\big(\|a|A\|+\|b|A\|\big)$ for all $a, b \in A$.

$\|\cdot|A\|$ is also called a *c-norm* and (iii) the *c-triangle inequality*. If (iii) holds with $c = 1$ then $\|\cdot|A\|$ is called a *norm*.

Definition 1.2.2. A quasi-normed space A is called *complete* if every Cauchy sequence converges to an element of A. Then, A is called *quasi-Banach space*. If A is normed, it is called *Banach space*.

Remark 1.2.1. Let A and B be quasi-normed spaces. We say that A is continuously embedded in B and write $A \hookrightarrow B$, if $A \subseteq B$ and there is a positive constant c such that $\|a|B\| \leq c\|a|A\|$ for all $a \in A$.

If $A \hookrightarrow B$ and $B \hookrightarrow A$, we say that the quasi-norms in A and B are equivalent and write $\|\cdot|A\| \sim \|\cdot|B\|$ or $\|a|A\| \sim \|a|B\|$.

We write $A = B$ if the spaces are equal as sets and the quasi-norms are equivalent.

The following definition of the *Lebesgue spaces* is taken from [12, p. 1].

Example 1.2.1. Let $\Omega \subseteq \mathbb{R}^n$. For all $0 < p < \infty$, the space $L_p(\Omega)$ is defined as the set of all complex-valued measurable functions on Ω such that

$$\int_\Omega |f(x)|^p \, dx < \infty.$$

The space $L_\infty(\Omega)$ consists of all measurable functions on Ω such that for some $B > 0$ the set

$$\{x \in \Omega \colon |f(x)| > B\}$$

has measure zero. We consider two functions equal if they are equal almost everywhere.

We put

$$\|f|L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

if $0 < p < \infty$ and

$$\|f|L_\infty(\Omega)\| = \inf\big\{B > 0 \colon \big|\{x \in \Omega \colon |f(x)| > B\}\big| = 0\big\}.$$

If $1 \leq p \leq \infty$, Minkowski's inequality

$$\|f + g|L_p(\Omega)\| \leq \|f|L_p(\Omega)\| + \|g|L_p(\Omega)\|$$

holds and $L_p(\Omega)$ is a normed space. For $0 < p < 1$ Minkowski's inequality does not hold, in fact, for $f, g \geq 0$, it is reversed (see [15]).

But, for $0 < p < 1$ we have

$$\|f + g|L_p(\Omega)\| \leq 2^{\frac{1-p}{p}} \big(\|f|L_p(\Omega)\| + \|g|L_p(\Omega)\| \big)$$

and so, $L_p(\Omega)$ is quasi-normed. For all $0 < p \le \infty$ one can show that $L_p(\Omega)$ is complete.

Lemma 1.2.1. *Let $\|\cdot|A\|$ be a c-norm and let γ be defined by $(2c)^\gamma = 2$. Then there is a norm $\|\cdot|A\|^*$ such that*

$$\|a|A\|^* \le \|a|A\|^\gamma \le 2\|a|A\|^*$$

for all $a \in A$.

Proof. A proof can be found in [3, pp. 59-60]. □

Lemma 1.2.2. *Let A be a c-normed space and let $(2c)^\gamma = 2$. Then A is complete if and only if*

$$\left(\sum_{j=0}^{\infty} \|a_j|A\|^\gamma\right)^{\frac{1}{\gamma}} < \infty$$

implies that $\sum_{j=0}^{\infty} a_j$ converges in A to an element $a \in A$ and

$$\left\|\sum_{j=0}^{\infty} a_j \middle| A\right\| \le \left(\sum_{j=0}^{\infty} \|a_j|A\|^\gamma\right)^{\frac{1}{\gamma}}.$$

Proof. From Lemma 1.2.1 follows

$$\left\|\sum_{j=n}^{m} a_j \middle| A\right\|^\gamma \le 2\left\|\sum_{j=n}^{m} a_j \middle| A\right\|^* \le 2\sum_{j=n}^{m} \|a_j|A\|^* \le 2\sum_{j=n}^{m} \|a_j|A\|^\gamma.$$

Therefore $\sum_{j=1}^{n} a_j$ is a Cauchy sequence and, because of the completeness of A, convergent in A. □

1.3 Non-increasing rearrangement and Lorentz spaces

Definition 1.3.1. Let $\Omega \subseteq \mathbb{R}^n$ and f be a measurable function on Ω. The function $d_f \colon [0,\infty) \to [0,\infty]$ is defined by

$$d_f(t) = \left|\{x \in \Omega \colon |f(x)| > t\}\right|.$$

We call d_f *distribution function* of f.

The following lemma is taken from [12, p. 4]. It shows that the distribution function contains all information about f that determines its L_p quasi-norm.

Lemma 1.3.1. *For $f \in L_p(\Omega)$, $0 < p < \infty$, we have*

$$\|f|L_p(\Omega)\|^p = p\int_0^\infty t^{p-1} d_f(t)\, dt.$$

Proof. We have

$$\begin{aligned} p\int_0^\infty t^{p-1} d_f(t)\, dt &= p\int_0^\infty t^{p-1}\int_\Omega \chi_{\{x\colon |f(x)|>t\}}(y)\, dy\, dt \\ &= \int_\Omega \int_0^{|f(y)|} pt^{p-1}\, dt\, dy \\ &= \int_\Omega |f(y)|^p\, dy. \end{aligned}$$

□

For a measurable function f we now want to construct a function f^* that has the same distribution function as f (f and f^* then are *equidistributed*).

Definition 1.3.2. Let f be a complex-valued function on Ω. We define the function $f^*\colon [0,\infty) \to [0,\infty]$ by

$$f^*(t) = \inf\big\{s > 0\colon d_f(s) \le t\big\},$$

where $\inf \emptyset = \infty$. The function f^* is called *non-increasing rearrangement* of f.

Lemma 1.3.2. *Let f be a measurable function on Ω. Then*

(i) $d_f(0) = |\operatorname{supp} f|$,

(ii) *f^* is non-increasing and right continuous,*

(iii) *if $|\Omega| < \infty$, then f^* is supported in $[0, |\Omega|]$,*

(iv) $d_f = d_{f^*}$,

(v) *if $0 < p < \infty$, then* $\|f|L_p(\Omega)\| = \left(\int_0^\infty f^*(t)^p\, dt\right)^{\frac{1}{p}}$,

(vi) $\|f|L_\infty(\Omega)\| = f^*(0)$,

(vii) *if $f^*(t) < \infty$, then $d_f\big(f^*(t)\big) \le t$.*

Proof. (i), (ii),(iii), and (vi) are immediate consequences from the definitions. For a proof of (iv) and (vii), see [12, p. 48]. (v) follows from Lemma 1.3.1 and (iv). □

With the help of the non-increasing rearrangement we can define the *Lorentz spaces.*

Definition 1.3.3. Let $0 < p, q \le \infty$ and f be a measurable function on Ω. If $0 < q < \infty$, we put

$$\|f|L_{p,q}(\Omega)\| = \left(\int_0^\infty \left\{t^{\frac{1}{p}} f^*(t)\right\}^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

and if $q = \infty$ we put

$$\|f|L_{p,\infty}(\Omega)\| = \sup_{t>0} t^{\frac{1}{p}} f^*(t).$$

The *Lorentz space* $L_{p,q}(\Omega)$ is the set of all f with $\|f|L_{p,q}(\Omega)\| < \infty$. Again, functions that are equal almost everywhere are considered equal.

Remark 1.3.1. For $0 < p \le \infty$ we have that $L_{p,p}(\Omega) = L_p(\Omega)$. This follows from Lemma 1.3.2 (iv). If $0 < q < \infty$, then the space $L_{\infty,q}(\Omega)$ consists only of the zero function.

The functional $\|f|L_{p,q}(\Omega)\|$ does not satisfy the triangle inequality, but the c-triangle inequality with $c = 2^{\frac{1}{p}} \max\left\{1, 2^{\frac{1-q}{q}}\right\}$. What is more, $L_{p,q}(\Omega)$ is a quasi-Banach space for all $0 < p, q \le \infty$. For a proof, see [12, p. 50].

Remark 1.3.2. If $1 < p < \infty$ and $1 \le q < \infty$, then $L_{p,q}(\Omega)$ can be equipped with a norm. Put

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds$$

for all $t > 0$ and

$$\|f|L_{p,q}(\Omega)\|' = \left(\int_0^\infty \left\{t^{\frac{1}{p}} f^{**}(t)\right\}^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

Then $\|\cdot|L_{p,q}(\Omega)\|'$ is a norm and it is equivalent to $\|\cdot|L_{p,q}(\Omega)\|$. See [3, p. 16].

For the following lemma, see [2, p. 217].

Lemma 1.3.3. (i) *Let* $0 < p \le \infty$ *and* $0 < q_1 < q_2 \le \infty$. *Then*

$$L_{p,q_1}(\Omega) \hookrightarrow L_{p,q_2}(\Omega).$$

(ii) *Suppose* $|\Omega| < \infty$ *and let* $0 < p_1 < p_2 \leq \infty$ *and* $0 < q_1, q_2 \leq \infty$. *Then*

$$L_{p_2,q_2}(\Omega) \hookrightarrow L_{p_1,q_1}(\Omega).$$

Now we generalize the Lorentz spaces by adding a logarithmic term. See [1, pp. 21-22,29].

Definition 1.3.4. Let $\Omega \subseteq \mathbb{R}^n$. Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. Then the *Lorentz-Zygmund space* $L_{p,q}(\log L)_\alpha(\Omega)$ consists of all measurable functions f on Ω for which

$$\|f|L_{p,q}(\log L)_\alpha(\Omega)\| = \begin{cases} \left(\int_0^\infty \left\{t^{\frac{1}{p}}(1+|\log t|)^\alpha f^*(t)\right\}^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty, \\ \sup\limits_{t>0}\left\{t^{\frac{1}{p}}(1+|\log t|)^\alpha f^*(t)\right\}, & q = \infty, \end{cases}$$

is finite.

If $p = q$, the spaces are called *Zygmund spaces* and are denoted by $L_p(\log L)_\alpha(\Omega)$.

Remark 1.3.3. For all $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$, the space $L_{p,q}(\log L)_\alpha(\Omega)$ is a quasi-Banach space. As in the case of Lorentz spaces, for $1 < p \leq \infty$ and $1 \leq q \leq \infty$, if we replace f^* by f^{**}, we get an equivalent norm. See [1, p. 30].

Next, we establish inclusion relations found in [1, p. 31].

Lemma 1.3.4. *Let* $|\Omega| < \infty$.

(i) *Let* $0 < p_1 < p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$, *and* $\alpha, \beta \in \mathbb{R}$. *Then*

$$L_{p_2,q_2}(\log L)_\beta(\Omega) \hookrightarrow L_{p_1,q_1}(\log L)_\alpha(\Omega).$$

(ii) *Let* $0 < p \leq \infty$, $0 < q_1, q_2 \leq \infty$, *and* $\alpha, \beta \in \mathbb{R}$ *such that either*

$$q_1 \leq q_2 \qquad \text{and} \qquad \alpha \geq \beta$$

or

$$q_1 > q_2 \qquad \text{and} \qquad \alpha + \frac{1}{q_1} > \beta + \frac{1}{q_2}.$$

Then

$$L_{p,q_1}(\log L)_\alpha(\Omega) \hookrightarrow L_{p,q_2}(\log L)_\beta(\Omega).$$

One can generalize the Lorentz-Zygmund spaces even further by replacing the logarithmic term in the definition by an appropriate function. Such functions will be introduced in the next section.

1.4 Slowly varying functions and Lorentz-Karamata spaces

The definitions and assertions in this section are taken from [11, pp. 87-89].

Definition 1.4.1. A nonnegative measurable function b on $(0, \infty)$ is said to be *slowly varying* if, for each $\varepsilon > 0$, there are nonnegative measurable functions g_ε and $g_{-\varepsilon}$ such that

(i) g_ε is non-decreasing and equivalent to $t^\varepsilon b(t)$ and

(ii) $g_{-\varepsilon}$ is non-increasing and equivalent to $t^{-\varepsilon} b(t)$.

By SV we denote the set of all slowly varying functions.

Theorem 1.4.1. *Let b, b_1, and b_2 be slowly varying functions.*

(i) *Then $b_1 b_2$, $b(1/t)$, b^r, $r \in \mathbb{R}$, and $b\big(t^p\big)$, $p > 0$, are slowly varying.*

(ii) *For $\varepsilon > 0$ and $\kappa > 0$ there are positive constants c_ε and C_ε such that*

$$c_\varepsilon \min(\kappa^{-\varepsilon}, \kappa^\varepsilon) b(t) \leq b(\kappa t) \leq C_\varepsilon \max(\kappa^\varepsilon, \kappa^{-\varepsilon}) b(t)$$

for every $t > 0$.

(iii) *For $\alpha > 0$ and $0 < q \leq \infty$ we have*

$$\left(\int_0^t \big\{ \tau^\alpha b(\tau) \big\}^q \frac{d\tau}{\tau} \right)^{\frac{1}{q}} \sim t^\alpha b(t)$$

and

$$\left(\int_t^\infty \big\{ \tau^{-\alpha} b(\tau) \big\}^q \frac{d\tau}{\tau} \right)^{\frac{1}{q}} \sim t^{-\alpha} b(t).$$

(iv) *Let $|\delta| \in (0, 1)$ and $d \in SV$. Then $d\big(t^\delta b(t)\big)$ is slowly varying.*

Proof. Let b, b_1, and b_2 be in SV and let $g_{\pm\varepsilon}$, $g^1_{\pm\varepsilon}$, and $g^2_{\pm\varepsilon}$ be the corresponding functions from Definition 1.4.1.

(i) The equivalences $t^\varepsilon b_1(t) b_2(t) \sim g^1_{\varepsilon/2} g^2_{\varepsilon/2}$ and $t^{-\varepsilon} b_1(t) b_2(t) \sim g^1_{-\varepsilon/2} g^2_{-\varepsilon/2}$ prove that $b_1 b_2 \in SV$.

Let $r \in \mathbb{R}$. If $r > 0$, then

$$t^\varepsilon b^r(t) = \big(t^{\varepsilon/r} b(t)\big)^r \sim \big(g_{\varepsilon/r}(t)\big)^r,$$

which is non-decreasing. If $r < 0$, then

$$t^{\varepsilon} b^{r}(t) \sim \big(g_{-\varepsilon/|r|}(t)\big)^{r},$$

which is also non-decreasing. The equivalence with non-increasing functions works in the same way. So, $b^r \in SV$.

If we replace t by $\frac{1}{t}$ in the equivalences

$$t^{\varepsilon} b(t) \sim g_{\varepsilon} \qquad \text{and} \qquad t^{\varepsilon} b(t) \sim g_{\varepsilon},$$

we get

$$t^{-\varepsilon} b\big(\tfrac{1}{t}\big) \sim g_{\varepsilon}\big(\tfrac{1}{t}\big) \qquad \text{and} \qquad t^{\varepsilon} b\big(\tfrac{1}{t}\big) \sim g_{-\varepsilon}\big(\tfrac{1}{t}\big).$$

It follows that $b\big(\frac{1}{t}\big) \in SV$.

Let $\varepsilon > 0$. It holds

$$t^{\varepsilon} b\big(t^{p}\big) = t^{\varepsilon-\delta p}\big[t^{p}\big]^{\delta} b\big(t^{p}\big).$$

Thus, if we choose δ sufficiently small and take into account that $b \in SV$, we can find a non-decreasing function that is equivalent to $t^{\varepsilon} b\big(t^{p}\big)$. Analogously, we see that the non-increasing part of the definition is satisfied as well.

(ii) For each $\varepsilon > 0$, $\kappa > 0$, and $t > 0$ we can write $b(\kappa t) = (\kappa t)^{-\varepsilon}(\kappa t)^{\varepsilon} b(\kappa t)$ to see that

$$b(\kappa t) \sim (\kappa t)^{-\varepsilon} g_{\varepsilon}(\kappa t) \quad \text{and} \quad b(\kappa t) \sim (\kappa t)^{\varepsilon} g_{-\varepsilon}(\kappa t).$$

If $\kappa \in (0,1)$, then

$$b(\kappa t) \leq C_{\varepsilon}(\kappa t)^{-\varepsilon} g_{\varepsilon}(t) \leq C_{\varepsilon}(\kappa t)^{-\varepsilon} t^{\varepsilon} b(t) = \kappa^{-\varepsilon} b(t),$$

and if $\kappa > 1$, then

$$b(\kappa t) \leq C_{\varepsilon}(\kappa t)^{\varepsilon} g_{-\varepsilon}(t) \leq C_{\varepsilon}(\kappa t)^{\varepsilon} t^{-\varepsilon} b(t) = \kappa^{\varepsilon} b(t).$$

The lower estimate can be proven analogously.

(iii) Let $\alpha > 0$, $0 < q \leq \infty$, and $t > 0$. By a simple calculation it follows that

$$t^{\alpha} \sim \left(\int_{t/2}^{t} \tau^{\alpha q}\, \frac{d\tau}{\tau}\right)^{\frac{1}{q}} \sim \left(\int_{0}^{t} \tau^{\alpha q}\, \frac{d\tau}{\tau}\right)^{\frac{1}{q}}.$$

If $\tau \in (t/2, t)$, then, using (ii) with $\varepsilon = 1$,

$$b(\tau) = b(\tfrac{\tau}{t}t) \geq c \min\Big(\frac{t}{\tau}, \frac{\tau}{t}\Big) b(t) \geq c\frac{\tau b(t)}{t} \geq c\frac{b(t)}{2}.$$

Therefrom and from Definition 1.4.1 it follows

$$\begin{aligned}
t^\alpha b(t) &\sim \Big(\int_{t/2}^t \tau^{-\alpha q}\,\frac{d\tau}{\tau}\Big)^{\frac{1}{q}} b(t) \lesssim \Big(\int_0^t \big\{\tau^\alpha b(\tau)\big\}^q\,\frac{d\tau}{\tau}\Big)^{\frac{1}{q}} \\
&\sim \Big(\int_0^t \big\{\tau^\alpha \tau^{-\alpha/2} g_{\alpha/2}(\tau)\big\}^q\,\frac{d\tau}{\tau}\Big)^{\frac{1}{q}} \leq g_{\alpha/2}(t)\Big(\int_0^t \big\{\tau^{\alpha/2}\big\}^q\,\frac{d\tau}{\tau}\Big)^{\frac{1}{q}} \\
&\sim g_{\alpha/2}(t) t^{\alpha/2} \sim t^\alpha b(t).
\end{aligned}$$

To prove the second equivalence we use

$$t^{-\alpha} \sim \Big(\int_t^{2t} \tau^{-\alpha q}\,\frac{d\tau}{\tau}\Big)^{\frac{1}{q}} \sim \Big(\int_t^\infty \tau^{-\alpha q}\,\frac{d\tau}{\tau}\Big)^{\frac{1}{q}}$$

and

$$b(\tau) \gtrsim \frac{b(t)}{2}$$

with $\tau \in (t, 2t)$ and follow the same arguments.

(iv) At first, assume that $0 < \delta < 1$. Then, for every $\varepsilon > 0$,

$$t^\varepsilon d\big(t^\delta b(t)\big) = t^{\varepsilon(1-\delta)} b(t)^{-\varepsilon}\ \big[t^\delta b(t)\big]^\varepsilon d\big(t^\delta b(t)\big)$$

is equivalent to a non-decreasing function, because $b(t)^{-\varepsilon}$ is slowly varying as well and $\varepsilon(1-\delta)$ is positive. Analogously we see that $t^{-\varepsilon} d\big(t^\delta b(t)\big)$ is equivalent to a non-increasing function. Now let $-1 < \delta < 0$. Then, according to the first step and (i), $d\big(t^{-\delta} b(t)^{-1}\big)$ is slowly varying. Consequently, again by (i), $d\big(t^\delta b(t)\big) \in SV$. □

Remark 1.4.1. If $b \in SV$, we can choose the functions given on the left-hand side in Theorem 1.4.1 (iii) as g_α and $g_{-\alpha}$. That means in particular that in Definition 1.4.1 we can assume without loss of generality that g_ε and $g_{-\varepsilon}$ are continuous.

Example 1.4.1. We put

$$\begin{aligned}
l_1(t) &:= 1 + |\log t|, \\
l_2(t) &:= 1 + \big|\log\big(1 + |\log t|\big)\big|, \text{ and} \\
l_{i+1}(t) &:= 1 + |\log(l_i)|, \quad \text{for } i \in \mathbb{N}.
\end{aligned}$$

Let $N \in \mathbb{N}$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$. Then

$$\lambda_{\bar{\alpha}}(t) := \prod_{i=1}^{N} l_i(t)^{\alpha_i}$$

is slowly varying.

Proof. We put $\tilde{l}_1(t) := 1+|\ln(t)|$ and $\tilde{l}_{i+1}(t) := 1+|\ln(\tilde{l}_i)|$ for $i \in \mathbb{N}$. For $t > 0$, we put $f(t) := t^\varepsilon \tilde{l}_i(t)$. Now, for each $\varepsilon > 0$ we have to find a monotonically non-decreasing function g_ε that is equivalent to f. By differentiation, we get for $t \neq 1$

$$\begin{aligned} f'(t) &= \varepsilon t^{\varepsilon-1}\tilde{l}_i(t) + t^\varepsilon \tilde{l}_i'(t) \\ &= \begin{cases} \varepsilon t^{\varepsilon-1}\tilde{l}_i(t) + t^\varepsilon \left[\tilde{l}_{i-1}(t)\tilde{l}_{i-2}(t)\cdots\tilde{l}_1(t)t\right]^{-1} & \text{if } t > 1, \\ \varepsilon t^{\varepsilon-1}\tilde{l}_i(t) - t^\varepsilon \left[\tilde{l}_{i-1}(t)\tilde{l}_{i-2}(t)\cdots\tilde{l}_1(t)t\right]^{-1} & \text{if } 0 < t < 1, \end{cases} \\ &= \begin{cases} \varepsilon t^{\varepsilon-1}\tilde{l}_i(t) + t^\varepsilon \left[\tilde{l}_{i-1}(t)\tilde{l}_{i-2}(t)\cdots\tilde{l}_1(t)t\right]^{-1} & \text{if } t > 1, \\ t^{\varepsilon-1}\dfrac{\varepsilon\tilde{l}_1(t)\cdots\tilde{l}_i(t) - 1}{\tilde{l}_1(t)\cdots\tilde{l}_{i-1}(t)} & \text{if } 0 < t < 1. \end{cases} \end{aligned}$$

Consequently, if $\varepsilon \geq 1$, we can put $g_\varepsilon(t) := f(t)$ because f is monotonically increasing in this case.

Suppose now that $0 < \varepsilon < 1$. We see that f is strictly monotonically increasing on $(1, \infty)$. In $(0, 1)$ one finds exactly one t_0, such that $f'(t_0) = 0$ and f is strictly increasing on $(0, t_0)$ and decreasing on $(t_0, 1)$. So

$$g_\varepsilon(t) := \begin{cases} f(t) & \text{if } 0 < t < t_0, \\ f(t_0) & \text{if } t_0 \leq t < 1/t_0, \\ f(t) & \text{if } t \geq 1/t_0 \end{cases}$$

is monotonically non-decreasing and equivalent to f.

In a similar way one can show that there is a non-increasing function equivalent to $t^{-\varepsilon}\tilde{l}_i(t)$ for every $\varepsilon > 0$. We have shown that $\tilde{l}_i$ is slowly varying. Consequently, l_i is slowly varying for each $i \in \mathbb{N}$ and with the help of Theorem 1.4.1 (i) it follows that $\lambda_{\bar{\alpha}} \in SV$. □

Definition 1.4.2. Let $\Omega \subseteq \mathbb{R}^n$ and let $0 < p, q \leq \infty$ and $b \in SV$. The *Lorentz-Karamata space* $L_{p,q;b}(\Omega)$ is the set of all measurable functions on Ω such that, for

$q < \infty$,

$$\|f|L_{p,q;b}(\Omega)\| = \left(\int_0^\infty \left\{ t^{\frac{1}{p}} b(t) f^*(t) \right\}^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$

In the case $q = \infty$ we proceed as we have done in the definition of the Lorentz and Lorentz-Zygmund spaces.

For this definition we refer to [6, p. 112] and [11, pp. 97-98]. In [6, p. 112 ff.] some interesting properties of these spaces are proved. For example, as in the case of Lorentz and Lorentz-Zygmund spaces, if $1 < p \leq \infty$ and $1 \leq q \leq \infty$, we get an equivalent norm in $L_{p,q;b}$ if we replace f^* by f^{**}.

Example 1.4.2. If $b(t) \equiv 1$, then $L_{p,q;b}(\Omega)$ coincides with the Lorentz space $L_{p,q}(\Omega)$. If $b(t) = (1 + |\log t|)^\alpha$ with $\alpha \in \mathbb{R}$, then $L_{p,q;b}(\Omega)$ equals the Lorentz-Zygmund space $L_{p,q}(\log L)_\alpha(\Omega)$.

Example 1.4.3. Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$. If we put $b(t) = \lambda_{\bar{\alpha}}$, where $\lambda_{\bar{\alpha}}$ has the meaning of Example 1.4.1, we denote the outcoming spaces by

$$L_{p,q;b}(\Omega) = L_{p,\bar{\alpha},q}(\Omega) = L_{p,\alpha_1,\ldots,\alpha_N,q}(\Omega).$$

These spaces are called *generalized Lorentz-Zygmund spaces* and have been studied in [9], [7], [19], [8], [10].

If $q = p$ we denote these spaces by $L_{p,\bar{\alpha}}(\Omega) = L_{p,\alpha_1,\ldots,\alpha_N}(\Omega)$.

2 Interpolation

2.1 Classical Real Interpolation

This introductory section about interpolation is based on [3].

Let A_0 and A_1 be quasi-normed vector spaces. We call the couple $\bar{A} = (A_0, A_1)$ *compatible*, if there exists a Hausdorff topological vector space $\mathscr{H}$ such that A_0 and A_1 are continuously embedded in $\mathscr{H}$. Then we can form their sum and intersection, where the sum is given by

$$\Sigma(\bar{A}) = A_0 + A_1 = \{a \in \mathscr{H} : a = a_0 + a_1, a_0 \in A_0 \text{ and } a_1 \in A_1\}$$

and the intersection by

$$\Delta(\bar{A}) = A_0 \cap A_1.$$

We can equip these spaces with the quasi-norms

$$\|a|\Sigma(\bar{A})\| = \inf_{a=a_0+a_1} (\|a_0|A_0\| + \|a_1|A_1\|)$$

and

$$\|a|\Delta(\bar{A})\| = \max(\|a|A_0\|, \|a|A_1\|)$$

respectively.

Lemma 2.1.1. *If A_0 and A_1 are complete, so are $A_0 + A_1$ and $A_0 \cap A_1$.*

Proof. The proof is carried over from the Banach space version [3, p. 25] as indicated in [3, p. 63]. We use Lemma 1.2.2. Let c_i be the constant in the triangle inequality of A_i, $i = 0, 1$, and put $c = \max(c_0, c_1)$. Define γ by $(2c)^\gamma = 2$.

Assume that

$$\sum_{j=1}^{\infty} \|a_j|A_0 + A_1\|^\gamma < \infty.$$

Then, we can find a decomposition $a_j = a_j^0 + a_j^1$, such that

$$\|a_j^0|A_0\| + \|a_j^1|A_1\| \le 2\|a_j|A_0 + A_1\|.$$

It follows that

$$\sum_{j=1}^{\infty} \|a_j^0|A_0\|^\gamma < \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} \|a_j^1|A_1\|^\gamma < \infty.$$

Because A_0 and A_1 are complete, $\sum_j a_j^0$ converges in A_0 and $\sum_j a_j^1$ converges in A_1. Put $a^0 = \sum_j a_j^0$, $a^1 = \sum_j a_j^1$, and $a = a^0 + a^1$. Then $a \in A_0 + A_1$ and

$$\Big\|a - \sum_{j=1}^{k} a_j \Big| A_0 + A_1\Big\| \leq \Big\|a^0 - \sum_{j=1}^{k} a_j^0 \Big| A_0\Big\| + \Big\|a^1 - \sum_{j=1}^{k} a_j^1 \Big| A_1\Big\|.$$

Consequently a_j converges in $A_0 + A_1$ to a. We proved that $A_0 + A_1$ is complete.

Now, let us consider the completeness of $A_0 \cap A_1$. Let a_j be a Cauchy sequence in $A_0 \cap A_1$. Then a_j is a Cauchy sequence in A_0 and in A_1, respectively, and since they are complete, there are elements $a_i \in A_i$, $i = 0, 1$, to which a_j converges. Because A_0 and A_1 are continuously embedded in a Hausdorff space, we have $a_0 = a_1$. So, a_j converges in $A_0 \cap A_1$ to $a := a_0 = a_1$. □

Let $\mathscr{L}(A, B)$ be the space of all bounded linear Operators from A to B, where A and B are quasi-normed linear spaces. If $A = B$ we write $\mathscr{L}(A)$ instead of $\mathscr{L}(A, A)$.

For two compatible couples $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ we write

$$T \in \mathscr{L}(\bar{A}, \bar{B}),$$

if $T \in \mathscr{L}\big(\Sigma(\bar{A}), \Sigma(\bar{B})\big)$ such that $T_{A_0} \in \mathscr{L}(A_0, B_0)$ and $T_{A_1} \in \mathscr{L}(A_1, B_1)$. Here T_C denotes the restriction of T to the space C, although in the sequel we will often call the restriction of an operator by the same symbol as the original operator. We will write $\mathscr{L}(\bar{A})$ instead of $\mathscr{L}(\bar{A}, \bar{A})$.

Definition 2.1.1. Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two compatible couples of quasi-normed spaces.

(i) A quasi-normed space A will be called *intermediate space* with respect to $\bar{A}$, if

$$\Delta(\bar{A}) \hookrightarrow A \hookrightarrow \Sigma(\bar{A}).$$

(ii) An intermediate space A is called *interpolation space* with respect to $\bar{A}$, if for every $T \in \mathscr{L}(\bar{A})$ the restriction of T to A is in $\mathscr{L}(A)$.

(iii) Let A and B be intermediate spaces with respect to $\bar{A}$ and $\bar{B}$ respectively. Then we say that A and B are *interpolation spaces* with respect to $\bar{A}$ and $\bar{B}$, if $T \in \mathscr{L}(\bar{A}, \bar{B})$ implies that $T \in \mathscr{L}(A, B)$.

It follows, that $\Sigma(\bar{A})$ and $\Delta(\bar{A})$ are interpolation spaces with respect to $\bar{A}$. Next we vary the quasi-norms in these spaces by a parameter t.

Definition 2.1.2. Let $\bar{A} = (A_0, A_1)$ be a compatible couple of quasi-normed spaces and let $t > 0$. We put

$$K(t, a) = K(t, a; \bar{A}) = \inf_{a=a_0+a_1} \big(\|a_0|A_0\| + t\|a_1|A_1\|\big)$$

for $a \in \Sigma(\bar{A})$ and

$$J(t, a) = J(t, a; \bar{A}) = \max\big(\|a|A_0\|, t\|a|A_1\|\big)$$

for $a \in \Delta(\bar{A})$.

$K(t, a)$ is called *Peetre's K-functional* and $J(t, a)$ *Peetre's J-functional.*

For applications of Interpolation Theory to concrete function spaces later on, we will need the K-functional for Lebesgue spaces.

Example 2.1.1. Let $0 < r < \infty$ and $f \in L_r(\Omega) + L_\infty(\Omega)$. Then

$$K(t, f; L_r, L_\infty) \sim \left(\int_0^{t^r} f^*(s)^r \, \frac{ds}{s}\right)^{\frac{1}{r}}.$$

If $r = 1$ we have even equality.

Proof. A proof can be found in [3, pp. 109-110]. See also [2, pp. 74,75,298]. □

Lemma 2.1.2. *For any $a \in A_0 + A_1$, $K(t, a)$ is a positive and increasing function of t. For all $s, t > 0$ holds*

$$K(t, a) \le \max(1, t/s) K(s, a),$$

which implies

$$\min(1, t)\|a|A_0 + A_1\| \le K(t, a) \le \max(1, t)\|a|A_0 + A_1\|.$$

Furthermore, for $a \in A_0 \cap A_1$, it holds

$$K(t, a) \le \min(1, t/s) J(s, a).$$

Proof. The proof is taken from [3, pp. 38,42].

To prove the first inequality we write

$$K(t,a) = \inf_{a=a_0+a_1}\Big(\|a_0|A_0\| + \frac{t}{s}s\|a_1|A_1\|\Big) \le \max\Big(1,\frac{t}{s}\Big)K(s,a).$$

For the last inequality take $a \in A_0 \cap A_1$. Then

$$K(t,a) \le \|a|A_0\| \le J(s,a)$$

and

$$K(t,a) \le \frac{t}{s}s\|a|A_1\| \le \frac{t}{s}J(s,a).$$

□

Lemma 2.1.3 (The fundamental lemma of interpolation theory)**.** *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces, where A_0 is c_0-normed and A_1 is c_1-normed. Let $a \in A_0 + A_1$ with*

$$\lim_{t\to 0} K(t,a) = 0 \quad \textit{and} \quad \lim_{t\to\infty}\frac{K(t,a)}{t} = 0.$$

Then, for all $\varepsilon > 0$, there is a sequence $(a_m)_m \subseteq A_0 \cap A_1$ such that

$$a = \sum_{m=-\infty}^{\infty} a_m \quad \textit{(convergence in } A_0 + A_1\textit{)}$$

and

$$J(2^m, a_m) \le (3\max(c_0,c_1) + \varepsilon)K(2^m, a).$$

Proof. See [3, pp. 45-46]. Take $a \in A_0 + A_1$ and let $\varepsilon > 0$. For every $j \in \mathbb{Z}$ there is a decomposition $a = a_{0,j} + a_{1,j}$ such that

$$\|a_{0,j}|A_0\| + 2^j\|a_{1,j}|A_1\| \le (1+\varepsilon)K(2^j, a). \tag{2.1.1}$$

It follows

$$\lim_{j\to-\infty}\|a_{0,j}|A_0\| = 0 \qquad \text{and} \qquad \lim_{j\to\infty}\|a_{1,j}|A_1\| = 0.$$

Put

$$u_j = a_{0,j} - a_{0,j-1} = a_{1,j-1} - a_{1,j}.$$

Then $u_j \in A_0 \cap A_1$ and

$$a - \sum_{-N}^{M} u_j = a - a_{0,M} + a_{0,-N-1} = a_{0,-N-1} + a_{1,M}.$$

Therefore, with $c = \max(c_0, c_1)$, we have

$$K\Big(1, a - \sum_{-N}^{M} u_j\Big) \leq c\Big(\|a_{0,-N-1}|A_0\| + \|a_{1,M}|A_1\|\Big).$$

It follows that $a = \sum\limits_{j=-\infty}^{\infty} u_j$ in $A_0 + A_1$.

By (2.1.1) we get

$$\begin{aligned} J(2^j, u_j) &\leq c \max\big\{\|a_{0,j}|A_0\| + \|a_{0,j-1}|A_0\|, 2^j\|a_{1,j-1}|A_1\| + 2^j\|a_{1,j}|A_1\|\big\} \\ &\leq c(1+\varepsilon)\big[K(2^j, a) + 2K(2^{j-1}, a)\big] \\ &\leq c(1+\varepsilon)3K(2^j, a). \end{aligned}$$

□

Definition 2.1.3. Let (A_0, A_1) be a compatible couple of quasi-Banach spaces. Let $0 < \theta < 1$ and let $0 < q \leq \infty$.

(i) For $a \in A_0 + A_1$, we put

$$\|a|A_{\theta,q;K}\| = \left(\sum_{m=-\infty}^{\infty} \big(2^{-m\theta} K(2^m, a)\big)^q\right)^{\frac{1}{q}},$$

if $0 < q < \infty$ and

$$\|a|A_{\theta,\infty;K}\| = \sup_{m\in\mathbb{Z}} \big(2^{-m\theta} K(2^m, a)\big),$$

if $q = \infty$.

Then the space $A_{\theta,q;K}$ consists of all $a \in A_0 + A_1$ with $\|a|A_{\theta,q;K}\| < \infty$.

(ii) By $A_{\theta,q;J}$ we denote the space of all $a \in A_0 + A_1$ for which there is a representation $a = \sum\limits_{m=-\infty}^{\infty} a_m$ (convergence in $A_0 + A_1$) with $a_m \in A_0 \cap A_1$ such that

$$\left(\sum_{m=-\infty}^{\infty} \big(2^{-m\theta} J(2^m, a_m)\big)^q\right)^{\frac{1}{q}} < \infty \qquad (0 < q < \infty) \tag{2.1.2}$$

or

$$\sup_{m\in\mathbb{Z}} \big(2^{-m\theta} J(2^m, a_m)\big) < \infty \qquad (q = \infty).$$

We put

$$\|a|A_{\theta,q;J}\| = \inf\bigg(\sum_{m=-\infty}^{\infty} \big(2^{-m\theta} J(2^m, a_m)\big)^q\bigg)^{\frac{1}{q}},$$

where the infimum is taken over all representations $a = \sum\limits_{m=-\infty}^{\infty} a_m$ with (2.1.2). For $q = \infty$ we have to make the usual modification.

Remark 2.1.1. Usually, the space $A_{\theta,q;K}$ is defined by the quasi-norm

$$\bigg(\int_0^\infty \big(t^{-\theta} K(t,a)\big)^q \frac{dt}{t}\bigg)^{\frac{1}{q}}.$$

In the case of Banach spaces, one can introduce the space $A_{\theta,q;J}$ as the space of all $a \in A_0 + A_1$ for which there is a representation $a = \int\limits_0^\infty u(t)\,\frac{dt}{t}$ with $u(t) \in A_0 \cap A_1$ such that

$$\bigg(\int_0^\infty \big(t^{-\theta} J(t,u(t))\big)^q \frac{dt}{t}\bigg)^{\frac{1}{q}} < \infty.$$

This definition is equivalent to Definition 2.1.3.

The following assertions are consequences from the more general theorems proved in the sections below. Proofs can be found also in [3] and [21].

Theorem 2.1.1. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be compatible couples of quasi-Banach spaces. Let $0 < \theta < 1$ and let $0 < q \le \infty$. Then $A_{\theta,q;K}$ and $B_{\theta,q;K}$ are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$. It holds*

$$\|T|\mathscr{L}(A_{\theta,q;K}, B_{\theta,q;K})\| \le \|T|\mathscr{L}(A_0,B_0)\|^{1-\theta} \|T|\mathscr{L}(A_1,B_1)\|^{\theta}$$

for all $T \in \mathscr{L}(\bar{A}, \bar{B})$.

The same is true for J instead of K.

Theorem 2.1.2. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces, $0 < \theta < 1$, and $0 < q \le \infty$. Then*

$$A_{\theta,q;K} = A_{\theta,q;J}.$$

Because interpolation using the K or J method gives the same result, we will write from now on $A_{\theta,q}$ instead of $A_{\theta,q;K}$ or $A_{\theta,q;J}$.

Theorem 2.1.3. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces. Then the following assertions hold.*

(i) $(A_0, A_1)_{\theta,q} = (A_1, A_0)_{1-\theta,q}$ *for* $0 < \theta < 1$ *and* $0 < q \leq \infty$.

(ii) *If* $0 < \theta < 1$ *and* $0 < q \leq r \leq \infty$, *then* $A_{\theta,q} \hookrightarrow A_{\theta,r}$.

(iii) *If additionally* $A_0 \hookrightarrow A_1$, *then*

$$A_{\theta_0,p} \hookrightarrow A_{\theta_1,q}$$

holds for $0 < \theta_0 < \theta_1 < 1$ *and* $0 < p, q \leq \infty$.

Theorem 2.1.4. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces. For $i = 0, 1$ let $0 < q_i, q \leq \infty$ and $\theta_i, \eta \in (0, 1)$. Then, with $\theta = (1 - \eta)\theta_0 + \eta\theta_1$, it holds*

$$\left(A_{\theta_0,q_0}, A_{\theta_1,q_1}\right)_{\eta,q} = A_{\theta,q}.$$

Theorem 2.1.5. *Let $p_0, p_1 \in (0, \infty)$, $q_0, q_1, q \in (0, \infty]$, let $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. If $p_0 \neq p_1$, then*

$$(L_{p_0,q_0}, L_{p_1,q_1})_{\theta,q} = L_{p,q}.$$

If $p_0 = p_1 = p$, we must, in addition, assume that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ to get

$$(L_{p,q_0}, L_{p,q_1})_{\theta,q} = L_{p,q}.$$

2.2 The function classes $B_{\mathcal{K}}$ and B_{Ψ}

We define the function class $B_{\mathcal{K}}$, first introduced by T. F. Kalugina ([17]) in 1975. The following definition is due to J. Gustavsson (1978) who showed in [13] that it leads to the same function class as the more difficult definition from Kalugina.

Definition 2.2.1. The function $f\colon (0, \infty) \to (0, \infty)$ belongs to the function class $B_{\mathcal{K}}$, if and only if f satisfies the following conditions:

(i) f is continuous and non-decreasing

(ii) For every $s > 0$ holds $\bar{f}(s) := \sup_{t>0} \dfrac{f(st)}{f(t)} < \infty$

(iii) $\displaystyle\int_0^\infty \min\Big(1, \frac{1}{t}\Big) \bar{f}(t)\, \frac{dt}{t} < \infty$

Example 2.2.1. Let $\theta \in (0,1)$. Then the function $f(t) = t^\theta$ is in $B_\mathcal{K}$ and $\bar{f}(s) = s^\theta$.

The following fact was pointed out in [11, Rem. 2.3 (ii)].

Lemma 2.2.1. *Let b be a slowly varying function, $0 < q \le \infty$, $0 < \theta < 1$, and let*

$$f(t) = \left(\int_0^t \left\{ \tau^\theta b(\tau) \right\}^q \frac{d\tau}{\tau} \right)^{\frac{1}{q}}$$

for $t > 0$. Then $t^\theta b(t)$ is equivalent to f and $f \in B_\mathcal{K}$.

Proof. Let $b \in SV$ and $\theta \in (0,1)$. We already showed in Theorem 1.4.1 (iii) that $t^\theta b(t) \sim f$. Now we show $f \in B_\mathcal{K}$ by checking (i) - (iii) in Definition 2.2.1. It is clear that f is continuous and non-decreasing. Using Theorem 1.4.1 (ii) we get

$$\begin{aligned}
\bar{f}(s) &= \sup_{t>0} \left(\frac{\int_0^{st} \left\{ \tau^\theta b(\tau) \right\}^q \frac{d\tau}{\tau}}{\int_0^t \left\{ \tau^\theta b(\tau) \right\}^q \frac{d\tau}{\tau}} \right)^{\frac{1}{q}} \\
&= s^\theta \sup_{t>0} \left(\frac{\int_0^t \left\{ \tau^\theta b(s\tau) \right\}^q \frac{d\tau}{\tau}}{\int_0^t \left\{ \tau^\theta b(\tau) \right\}^q \frac{d\tau}{\tau}} \right)^{\frac{1}{q}} \\
&\lesssim s^\theta \max\left(s^\varepsilon, \frac{1}{s^\varepsilon} \right) < \infty
\end{aligned}$$

for every $s > 0$ and $\varepsilon > 0$.

To prove (iii) we put $\varepsilon = \min\{\theta, 1-\theta\}$ in the above estimate. Then

$$\begin{aligned}
&\int_0^\infty \min\big(1, 1/t\big) \bar{f}(t) \frac{dt}{t} \\
&\le c_\varepsilon \int_0^1 \max(t^\varepsilon, 1/t^\varepsilon) t^\theta \frac{dt}{t} + c_\varepsilon \int_1^\infty \frac{\max(t^\varepsilon, 1/t^\varepsilon)}{t} t^\theta \frac{dt}{t} \\
&= c_\varepsilon \int_0^1 \frac{dt}{t^{1+\varepsilon-\theta}} + c_\varepsilon \int_1^\infty \frac{dt}{t^{2-\varepsilon-\theta}} < \infty
\end{aligned}$$

□

Example 2.2.2. Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and $0 < \theta < 1$. Then

$$t^\theta (1 + |\log t|)^\alpha \sim \left(\int_0^t \left\{ \tau^\theta (1 + |\log \tau|)^\alpha \right\}^q \frac{d\tau}{\tau} \right)^{\frac{1}{q}} \in B_\mathcal{K}.$$

We give some important properties of functions in $B_\mathcal{K}$.

Theorem 2.2.1. *Let $f \in B_\mathcal{K}$ and let $\underline{f}(s) = \inf_{t>0} \dfrac{f(st)}{f(t)}$. It holds*

(i) $\underline{f}(s)\bar{f}\Big(\frac{1}{s}\Big) = 1$

(ii) $0 < \underline{f}(s)f(t) \leq f(st) \leq \bar{f}(s)f(t)$

(iii) $\bar{f}$ *and* $\underline{f}$ *are non-decreasing and* $\bar{f}(1) = \underline{f}(1) = 1$

(iv) $\bar{f}(st) \leq \bar{f}(s)\bar{f}(t)$

(v) $\lim_{s\to\infty} \frac{\bar{f}(s)}{s} = 0$ *and* $\lim_{s\to 0} \bar{f}(s) = 0$

Proof. The proofs are taken from [13, pp. 290-291]. (i) Let $s > 0$. Then

$$\bar{f}(1/s) = \sup_{t>0} \frac{f(t/s)}{f(t)} = \sup_{t>0} \frac{f(t)}{f(st)} = \frac{1}{\inf\limits_{t>0} \frac{f(st)}{f(t)}} = \frac{1}{\underline{f}(s)}.$$

(ii) From $\bar{f}(s) < \infty$ it follows by (i) that $\underline{f}(s) > 0$. Moreover

$$\underline{f}(s)f(t) \leq \frac{f(st)}{f(t)} f(t) \leq \bar{f}(s)f(t).$$

(iii) follows directly from the definition.
(iv) With the help of (ii) it follows

$$\bar{f}(st) = \sup_{r>0} \frac{f(str)}{f(r)} \leq \sup_{r>0} \frac{\bar{f}(s)f(tr)}{f(r)} = \bar{f}(s)\bar{f}(t).$$

(v) We use the monotonicity of $\bar{f}$ to get for $s > 0$

$$\int_s^\infty \frac{\bar{f}(t)}{t} \frac{dt}{t} \geq \bar{f}(s) \int_s^\infty \frac{dt}{t^2} = \frac{\bar{f}(s)}{s}$$

and

$$\int_s^{es} \bar{f}(t) \frac{dt}{t} \geq \bar{f}(s) \int_s^{es} \frac{dt}{t} = \bar{f}(s).$$

Now, by Definition 2.2.1 (iii) the desired assertions follow. □

Definition 2.2.2. A function $f\colon (0,\infty) \to (0,\infty)$ is called *submultiplicative*, if

$$f(st) \leq f(s)f(t)$$

for all $s, t \in (0,\infty)$ and $f(1) = 1$.

Example 2.2.3. As the above theorem states, for $f \in B_{\mathcal{K}}$ the function $\bar{f}$ is submultiplicative.

Lemma 2.2.2. *Let $f\colon (0,\infty) \to (0,\infty)$ be submultiplicative. Then f is bounded on each interval (x,y) with $0 < x < y < \infty$.*

Proof. The proof is analogous to the proof of the boundedness of subadditive functions in [16, p. 241]. For a measurable set $A \subseteq (0,\infty)$ let $\mu(A) = \int_A \frac{dt}{t}$. Observe that $\mu(A) < |A|$ if $a > 1$ for all $a \in A$.

First we prove that f is bounded on any interval (x,y) with $1 < x < y < \infty$. Suppose f is unbounded from above in (x,y). Then we can choose a sequence $(t_n)_n \subseteq (x,y)$ such that $f(t_n) \geq n^2$. For each $n \in \mathbb{N}$ let $E_n = \{t \in (1,y)\colon f(t) \geq n\}$.

Now, let $n \in \mathbb{N}$ and take $r \in (1,x)$. Then one can find $s \in (1,t_n)$ such that $rs = t_n$. It follows that $n^2 \leq f(t_n) \leq f(r)f(s)$. That means either $f(r) \geq n$ or $f(s) \geq n$ which implies $r \in E_n \cup \{\frac{t_n}{t}\colon t \in E_n\}$. Consequently $(1,x) \subseteq E_n \cup \{\frac{t_n}{t}\colon t \in E_n\}$.

Because of $\mu(E_n) = \mu\big(\frac{t_n}{E_n}\big)$ we now have $\mu(E_n) \geq \frac{\mu((1,x))}{2} = \frac{\log x}{2}$. It follows that $0 = |\bigcap_n E_n| \geq \frac{\log x}{2}$.

Secondly we show that f is bounded in $(\delta, 1+\delta)$ for any $\delta \in (0,1)$. Consider $\delta \in (0,1)$ fixed. Then for all $t \in (\delta, 1+\delta)$ we have

$$f(t) \leq f\Big(\frac{2t}{\delta}\Big) f\Big(\frac{\delta}{2}\Big) \leq C_\delta,$$

because of $2t/\delta \geq 2$ and f is bounded in (x,y), $1 < x < y$. □

For the following lemma we used [16, p. 244] and [14, p. 35].

Lemma 2.2.3. *Let f be a submultiplicative function and let*

$$\alpha := \sup_{0<t<1} \frac{\log f(t)}{\log t} \quad \textit{and} \quad \beta := \inf_{t>1} \frac{\log f(t)}{\log t}.$$

Then it follows that

(i) $-\infty < \alpha = \lim_{t\to 0} \frac{\log f(t)}{\log t} \leq \beta = \lim_{t\to\infty} \frac{\log f(t)}{\log t} < \infty$,

(ii) $\alpha = \sup\{p \in \mathbb{R}\colon$ *for some* $0 < r < 1$ *and all* $t \in (0,r)$ *it holds* $f(t) \leq t^p\}$,

(iii) $\beta = \inf\{p \in \mathbb{R}\colon$ *for some* $r > 1$ *and all* $t > r$ *it holds* $f(t) \leq t^p\}$.

Proof. (i) The proof is a modification of [16, p. 244], where subadditive functions are treated instead of submultiplicative ones.

If $t > 1$ it follows from $1 = f(1) \leq f(t)f(1/t)$, that

$$\frac{\log f(1/t)}{\log(1/t)} \leq \frac{\log f(t)}{\log t}.$$

This implies $-\infty < \alpha \leq \beta < \infty$.

We choose $b > 1$ such that $\frac{\log f(b)}{\log b} < \beta + \varepsilon$. Let $t > 1$ and choose $n = n(t) \in \mathbb{N}$ such that $b^n \leq t \leq b^{n+1}$. Then

$$\begin{aligned} \beta \leq \frac{\log f(t)}{\log t} &= \frac{\log f(b^n t/b^n)}{\log t} \leq \frac{\log f(b^n)}{\log t} + \frac{\log f(t/b^n)}{\log t} \\ &= \frac{\log b^n}{\log t}\frac{\log f(b)}{\log b} + \frac{\log f(t/b^n)}{\log t} \\ &< \frac{\log b^n}{\log t}(\beta+\varepsilon) + \frac{\log f(t/b^n)}{\log t}. \end{aligned} \tag{2.2.1}$$

Since $t/b^n \in [1, b]$ it follows from Lemma 2.2.2 that $f(t/b^n)$ is bounded. So the last expression in (2.2.1) tends to $\beta + \varepsilon$ as $t \to \infty$. It follows that $\lim\limits_{t\to\infty} \frac{\log f(t)}{\log t}$ exists and equals β.

By applying the above result to $f(1/t)$ we get:

$$\begin{aligned} \alpha = \sup_{0<t<1} \frac{\log f(t)}{\log t} &= -\inf_{0<t<1}\Big(-\frac{\log f(t)}{\log t}\Big) = -\inf_{t>1}\Big(-\frac{\log f(1/t)}{\log 1/t}\Big) \\ &= -\inf_{t>1}\frac{\log f(1/t)}{\log t} = -\lim_{t\to\infty}\frac{\log f(1/t)}{\log t} = \lim_{t\to\infty}\frac{\log f(1/t)}{\log 1/t} = \lim_{t\to 0}\frac{\log f(t)}{\log t}. \end{aligned}$$

(ii) Let $\varepsilon > 0$. Because $\alpha = \lim\limits_{t\to 0}\frac{\log f(t)}{\log t}$ there is an $r \in (0,1)$ such that $\alpha - \varepsilon < \frac{\log f(t)}{\log t} \leq \alpha$ applies to all $t \in (0, r)$. It follows that

$$t^\alpha \leq f(t) < t^{\alpha-\varepsilon}$$

for all $t \in (0, r)$. So α is the supremum of all $p \in \mathbb{R}$ with the property mentioned above.

(iii) Analogously to (ii) it follows that for all $\varepsilon > 0$ there is an $r > 1$ such that

$$t^\beta \leq f(t) < t^{\beta+\varepsilon}$$

holds for all $t \in (r, \infty)$. This proves (iii). □

Lemma 2.2.4. *Let $f \in B_{\mathcal{K}}$. Then*

(i) *There exist $\varepsilon > 0$ and $s_0 \in (0,1)$ such that $\bar{f}(s) \leq s^\varepsilon$ for $0 < s \leq s_0$.*

(ii) *There exist $\varepsilon > 0$ and $s_1 \in (1,\infty)$ such that $\bar{f}(s) \leq s^{1-\varepsilon}$ for $s \geq s_1$.*

(iii) *It holds*

$$\int_0^\infty \Big\{\min\Big(1, \frac{1}{t}\Big)\bar{f}(t)\Big\}^p \frac{dt}{t} < \infty$$

for any $p > 0$.

Proof. (i) In Theorem 2.2.1 (v) we showed that $\lim_{s\to 0} \bar{f}(s) = 0$. So, in particular there are $\tilde{\varepsilon} \in (0, 1)$ and $\tilde{s} \in (0, 1)$ such that $\bar{f}(\tilde{s}) \le \tilde{\varepsilon}$. We can conclude that

$$0 < \frac{\log \tilde{\varepsilon}}{\log \tilde{s}} \le \frac{\log \bar{f}(\tilde{s})}{\log \tilde{s}} \le \alpha,$$

where $\alpha = \sup_{0<s<1} \frac{\log \bar{f}(s)}{\log s}$. Because of Lemma 2.2.1 (ii) assertion (i) is proved.

(ii) Similarly to (i) we can conclude from $\lim_{s\to\infty} \frac{\bar{f}(s)}{s} = 0$ that there is an $\varepsilon > 0$ and an $\tilde{s} > 1$ such that $\bar{f}(\tilde{s}) \le \tilde{\varepsilon}\tilde{s}$. Consequently

$$\beta \le \frac{\log \bar{f}(s)}{\log s} \le \frac{\log \varepsilon}{\log s} + 1 < 1.$$

Now (ii) follows from 2.2.1 (iii).

(iii) Using (i) and (ii) we get

$$\begin{aligned}
&\int_0^\infty \Big\{\min\Big(1, \frac{1}{t}\Big)\bar{f}(t)\Big\}^p \frac{dt}{t} \\
\le\ &\int_0^{s_0} \frac{dt}{t^{1-\varepsilon p}} + \int_{s_0}^{s_1} \Big\{\min\Big(1, \frac{1}{t}\Big)\bar{f}(t)\Big\}^p \frac{dt}{t} + \int_{s_1}^\infty \frac{dt}{t^{1+\varepsilon p}} < \infty.
\end{aligned}$$

□

Now we define the class B_Ψ and prove basic properties as it has been done in [13, p. 292].

Definition 2.2.3. The function class B_Ψ consists of all continuously differentiable functions $f\colon (0, \infty) \to (0, \infty)$ such that

$$\sup_{t>0} \frac{tf'(t)}{f(t)} < 1 \qquad \text{and} \qquad \inf_{t>0} \frac{tf'(t)}{f(t)} > 0.$$

Example 2.2.4. For $0 < \theta < 1$ and $\alpha \in \mathbb{R}$ the function f with

$$f(t) = t^\theta \big(\ln(1 + t^\gamma)\big)^\alpha$$

belongs to B_Ψ, if $\gamma > 0$ is sufficiently small.

Proof. Because of

$$\frac{tf'(t)}{f(t)} = \theta + \frac{\alpha\gamma t^\gamma}{(1+t^\gamma)\ln(1+t^\gamma)}$$

we get

$$\sup_{t>0} \frac{tf'(t)}{f(t)} = \max(\theta, \theta + \alpha\gamma)$$

and

$$\inf_{t>0} \frac{tf'(t)}{f(t)} = \min(\theta, \theta + \alpha\gamma).$$

□

Lemma 2.2.5. (i) *It holds $B_\Psi \subseteq B_\mathcal{K}$.*

(ii) *For each $f \in B_\mathcal{K}$ there exists $g \in B_\Psi$ such that $f \sim g$.*

Proof. (i) Let $f \in B_\Psi$. Because of $f'(t) > 0$, Definition 2.2.1 (i) is satisfied. For $t, s > 0$ we define

$$h_t(s) = \frac{f(st)}{f(t)}$$

and put $p = \inf \frac{tf'(t)}{f(t)}$ and $q = \sup \frac{tf'(t)}{f(t)}$. Then

$$p \leq \frac{sh_t'(s)}{h_t(s)} = \frac{stf'(st)}{f(st)} \leq q.$$

It follows that

$$\frac{ps^{p-1}}{s^p} \leq \frac{h_t'(s)}{h_t(s)} \leq \frac{qs^{q-1}}{s^q}.$$

This implies

$$\left(\frac{h_t(s)}{s^p}\right)' \geq 0 \quad \text{and} \quad \left(\frac{h_t(s)}{s^q}\right)' \leq 0.$$

Because $h_t(1) = 1$ we can conclude $h_t(s) \leq s^p$, if $0 < s < 1$ and $h_t(s) \leq s^q$, if $s > 1$. Consequently, $\bar{f}(s) \leq \max(s^p, s^q)$. Therefrom we obtain (ii) and (iii) of Definition 2.2.1.

(ii) Let $f \in B_\mathcal{K}$ and put

$$g(s) = \int_0^\infty \min\Big(1, \frac{s}{t}\Big) f(t) \, \frac{dt}{t}.$$

It holds

$$f(s) \leq \int_s^\infty \frac{s}{t} f(t) \, \frac{dt}{t} \leq g(s)$$

and

$$g(s) = \int_0^\infty \min\Big(1, \frac{1}{t}\Big) f(st)\, \frac{dt}{t} \le f(s) \int_0^\infty \min\Big(1, \frac{1}{t}\Big) \bar{f}(t)\, \frac{dt}{t} = cf(s),$$

which proves that $f \sim g$. Since f is continuous and

$$g(s) = \int_0^s f(t)\, \frac{dt}{t} + s \int_s^\infty \frac{f(t)}{t} \frac{dt}{t}$$

we find

$$g'(s) = \int_s^\infty \frac{f(t)}{t} \frac{dt}{t}.$$

Consequently, g is continuously differentiable and from

$$f(s) \le s \int_s^\infty \frac{f(t)}{t} \frac{dt}{t} = \int_1^\infty \frac{f(st)}{t} \frac{dt}{t} \le f(s) \int_1^\infty \frac{\bar{f}(t)}{t} \frac{dt}{t} = c_1 f(s)$$

and

$$c_2 f(s) \le f(s) \int_0^1 \underline{f}(t)\, \frac{dt}{t} \le \int_0^s f(t)\, \frac{dt}{t} \le f(s) \int_0^1 \bar{f}(t)\, \frac{dt}{t} = c_3 f(s)$$

we get

$$1 + \frac{c_2}{c_1} \le 1 + \frac{\displaystyle\int_0^s f(t)\, \frac{dt}{t}}{\displaystyle s \int_s^\infty \frac{f(t)}{t} \frac{dt}{t}} \le 1 + c_3.$$

Now it follows that

$$\inf_{s>0} \frac{sg'(s)}{g(s)} > 0 \qquad \text{and} \qquad \sup_{s>0} \frac{sg'(s)}{g(s)} < 1.$$

□

2.3 Interpolation with function parameters

We now want to modify the real interpolation method by replacing t^θ by a more general function $\varrho\colon (0,\infty) \to (0,\infty)$. Several authors have done that and defined classes of admissible functions. Kalugina [17] and Gustavsson [13] used functions in $B_{\mathcal{K}}$ and in B_Ψ, Persson [20] defined $Q(0,1)$ and Gustavsson and Peetre [14] defined the class $\mathscr{P}^{+-}$. It turned out that all these classes of functions coincide in the following sense (see Lemma 2.2.5 and [20, Prop. 1.3]).

$$B_\Psi \subseteq B_\mathcal{K},$$

$$B_\Psi \subseteq Q(0,1) \subseteq \mathscr{P}^{+-},$$

and for a function ϱ in $B_\mathcal{K}$ or $\mathscr{P}^{+-}$ there is a function in B_Ψ, that is equivalent to ϱ.

Later on we also want to use functions of the form $\varrho(t) = t^\theta b(t)$, where $\theta \in (0,1)$ and $b \in SV$. We have seen in Lemma 2.2.1 that for each of those ϱ there is a function $f \in B_\mathcal{K}$ that is equivalent to ϱ. To cover all these types of function parameters we make the following definition.

Definition 2.3.1. We say that a function $\varrho\colon (0,\infty) \to (0,\infty)$ belongs to FP if and only if there is a function $f \in B_\Psi$ such that $\varrho \sim f$.

Definition 2.3.2. Let (A_0, A_1) be a compatible couple of quasi-Banach spaces. Let $\varrho \in FP$ and let $0 < q \leq \infty$.

(i) For $a \in A_0 + A_1$, we put

$$\|a|A_{\varrho,q;K}\| = \left(\sum_{m=-\infty}^{\infty} \frac{K(2^m,a)^q}{\varrho(2^m)^q} \right)^{\frac{1}{q}},$$

if $0 < q < \infty$ and

$$\|a|A_{\varrho,\infty;K}\| = \sup_{m\in\mathbb{Z}} \frac{K(2^m,a)}{\varrho(2^m)},$$

if $q = \infty$.

Then the space $A_{\varrho,q;K}$ consists of all $a \in A_0 + A_1$ with $\|a|A_{\varrho,q;K}\| < \infty$.

(ii) Let $A_{\varrho,q;J}$ be the space of all $a \in A_0 + A_1$ for which there is a representation $a = \sum\limits_{m=-\infty}^{\infty} a_m$ (convergence in $A_0 + A_1$) with $a_m \in A_0 \cap A_1$ such that

$$\left(\sum_{m=-\infty}^{\infty} \frac{J(2^m,a_m)^q}{\varrho(2^m)^q} \right)^{\frac{1}{q}} < \infty \qquad (0 < q < \infty) \tag{2.3.1}$$

or

$$\sup_{m\in\mathbb{Z}} \frac{J(2^m,a_m)}{\varrho(2^m)} < \infty \qquad (q = \infty).$$

For $0 < q < \infty$ we put

$$\|a|A_{\varrho,q;J}\| = \inf\Bigg(\sum_{m=-\infty}^{\infty} \frac{J(2^m, a_m)^q}{\varrho(2^m)^q}\Bigg)^{\frac{1}{q}},$$

where the infimum is taken over all representations $a = \sum_{m=-\infty}^{\infty} a_m$ satisfying (2.3.1).

For $q = \infty$ we put

$$\|a|A_{\varrho,\infty;J}\| = \inf \sup_{m\in\mathbb{Z}} \frac{J(2^m, a_m)}{\varrho(2^m)}.$$

Remark 2.3.1. If $\varrho_1, \varrho_2 \in FP$ with $\varrho_1 \sim \varrho_2$, then $\|a|A_{\varrho_1,q;K}\| \sim \|a|A_{\varrho_2,q;K}\|$ and $\|a|\varrho_1, q; J\| \sim \|a|A_{\varrho_2,q;J}\|$.

Next we show that the sum in the definition of the K-method can be replaced by an integral. We will not do that for the J-method, because we want to avoid integration of functions with values in a quasi-normed space.

Theorem 2.3.1. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces, let $\varrho \in FP$, and let $0 < q \leq \infty$. Then*

$$\|a|A_{\varrho,q;K}\| \sim \Bigg(\int_0^\infty \varrho(t)^{-q} K(t,a)^q \,\frac{dt}{t}\Bigg)^{\frac{1}{q}}.$$

Proof. Let $f \in B_{\mathcal{K}}$ such that $f \sim \varrho$. Using Lemma 2.1.2 and Theorem 2.2.1 (ii) we obtain the estimates

$$\begin{aligned}
\int_0^\infty f(t)^{-q} K(t,a)^q \,\frac{dt}{t} &= \sum_{m=-\infty}^{\infty} \int_{2^m}^{2^{m+1}} f(t)^{-q} K(t,a)^q \,\frac{dt}{t} \\
&\leq \log 2 \sum_{m=-\infty}^{\infty} f(2^m)^{-q} K(2\cdot 2^m, a)^q \\
&\leq 2^q \log 2 \sum_{m=-\infty}^{\infty} f(2^m)^{-q} K(2^m, a)^q
\end{aligned}$$

and

$$\int_0^\infty f(t)^{-q} K(t,a)^q \,\frac{dt}{t} = \sum_{m=-\infty}^{\infty} \int_{2^m}^{2^{m+1}} f(t)^{-q} K(t,a)^q \,\frac{dt}{t}$$

$$\geq \log 2 \sum_{m=-\infty}^{\infty} f(2\cdot 2^m)^{-q} K(2^m,a)^q$$

$$\geq \bar{f}(2)^{-q} \log 2 \sum_{m=-\infty}^{\infty} f(2^m)^{-q} K(2^m,a)^q.$$

□

Lemma 2.3.1. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces and let $\varrho \in FP$. Then*

(i) $K(s,a) \leq c\varrho(s)\|a|A_{\varrho,q;K}\|$ *for all* $s > 0$,

(ii) $\|a|A_{\varrho,q;K}\| \leq \dfrac{c}{\varrho(s)} J(s,a)$ *for all* $s > 0$,

(iii) $(A_0, A_1)_{\varrho,q;K} = (A_1, A_0)_{\varphi,q;K}$, *where* $\varphi(t) = t\varrho(1/t)$.

Proof. Let $f \in B_\Psi$ such that $\varrho \sim f$. We make use of Remark 2.3.1.

(i) We have $K(t,a) \geq \min(1, t/s)K(s,a)$ by Lemma 2.1.2. Together with the monotonicity of f we get

$$\|a|A_{f,q;K}\| \geq \left(\int_0^s f(t)^{-q} K(t,a)^q \,\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$\geq f(s)^{-1} K(s,a) \left(\int_0^s \min(1,t/s)^q \,\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$= f(s)^{-1} K(s,a) \frac{1}{q^{\frac{1}{q}}}.$$

(ii) Lemma 2.1.2 implies $K(s,a) \leq \min\big(1, \frac{s}{t}\big) J(t,a)$. It follows

$$\|a|A_{f,q;K}\| \leq \left(\int_0^\infty f(t)^{-q} \min\big(1, \frac{t}{s}\big)^q J(s,a)^q \,\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$= J(s,a) \frac{1}{f(s)} \left(\int_0^\infty \min(1,t)^q \frac{f(s)^q}{f(st)^q} \,\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$\leq J(s,a) \frac{1}{f(s)} \left(\int_0^\infty \Big\{\min\big(1, \frac{1}{t}\big) \bar{f}(t)\Big\}^q \,\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$= \frac{c}{f(s)} J(s,a).$$

The convergence of the integral has been proved in Lemma 2.2.4 (iii).

(iii) Put $g(t) = tf\left(\frac{1}{t}\right)$. We have $\dfrac{tg'(t)}{g(t)} = 1 - \dfrac{{}^1\!/_t f'({}^1\!/_t)}{f({}^1\!/_t)}$ and, consequently, $g \in B_\Psi$. Because of $\varphi \sim g$ we have $\varphi \in FP$. Using $K(t, a; A_0, A_1) = tK({}^1\!/_t, a; A_1, A_0)$ it follows

$$\begin{aligned}\|a|A_{f,q;K}\|^q &= \int_0^\infty \frac{t^q K({}^1\!/_t, a; A_0, A_1)^q}{t^q f({}^1\!/_t)^q} \frac{dt}{t} \\ &= \int_0^\infty \frac{K(t, a; A_1, A_0)^q}{g(t)^q} \frac{dt}{t}.\end{aligned}$$

□

Theorem 2.3.2 (Equivalence theorem). *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces, let $0 < q \leq \infty$ and let $\varrho \in FP$. Then*

$$A_{\varrho,q;K} = A_{\varrho,q;J}.$$

Remark 2.3.2. If we put $\varrho(t) = t^\theta$ for some $\theta \in (0, 1)$, then we get the classical Equivalence Theorem (Theorem 2.1.2).

Proof. Let $f \in B_\Psi$ such that $\varrho \sim f$. We first prove $A_{f,q;J} \hookrightarrow A_{f,q;K}$. Let $a = \sum_{m=-\infty}^{\infty} a_m \in A_{f,q;J}$. Recall that $K(t, a)$ is a c-norm on $A_0 + A_1$ for all $t > 0$. We can choose the constant c in the c-triangle inequality of $K(t, a)$ large, such that γ, defined by $(2c)^\gamma = 2$, is smaller than q. Therefore we have $p := \frac{q}{\gamma} > 1$. From Lemma 1.2.2 and Lemma 2.1.2 with $s = 2^m$ we obtain

$$\begin{aligned}K(t, a) &\leq \left(\sum_{m=-\infty}^{\infty} \big(K(t, a_m)\big)^\gamma \right)^{\frac{1}{\gamma}} \\ &\leq \left(\sum_{m=-\infty}^{\infty} \big(\min(1, t2^{-m}) J(2^m, a_m)\big)^\gamma \right)^{\frac{1}{\gamma}}.\end{aligned}$$

Now we put $t = 2^n$ and find

$$\begin{aligned}K(2^n, a) &\leq \left(\sum_{m=-\infty}^{\infty} \big(\min(1, 2^{n-m}) J(2^m, a_m)\big)^\gamma \right)^{\frac{1}{\gamma}} \\ &= \left(\sum_{m=-\infty}^{\infty} \big(\min(1, 2^m) J(2^{n-m}, a_{n-m})\big)^\gamma \right)^{\frac{1}{\gamma}}.\end{aligned}$$

Using Minkowski's inequality for infinite series (see [15, p. 123]) we obtain

$$\begin{aligned}
\|a|A_{f,q;K}\| &= \Big(\sum_{n=-\infty}^{\infty} f(2^n)^{-q} K(2^n,a)^q\Big)^{\frac{1}{q}} \\
&= \Big(\Big[\sum_{n=-\infty}^{\infty} \big(f(2^n)^{-\gamma} K(2^n,a)^{\gamma}\big)^p\Big]^{\frac{1}{p}}\Big)^{\frac{1}{\gamma}} \\
&\le \Big(\Big[\sum_{n=-\infty}^{\infty} \Big(f(2^n)^{-\gamma} \sum_{m=-\infty}^{\infty} \min(1,2^m)^{\gamma} J(2^{n-m},a_{n-m})^{\gamma}\Big)^p\Big]^{\frac{1}{p}}\Big)^{\frac{1}{\gamma}} \\
&\le \Big(\sum_{m=-\infty}^{\infty} \min(1,2^m)^{\gamma}\Big[\sum_{n=-\infty}^{\infty}\Big(\frac{J(2^{n-m},a_{n-m})}{f(2^n)}\Big)^{\gamma p}\Big]^{\frac{1}{p}}\Big)^{\frac{1}{\gamma}} \\
&\le \Big(\sum_{m=-\infty}^{\infty} \min(1,2^m)^{\gamma}\Big[\sum_{n=-\infty}^{\infty}\Big(\frac{J(2^{n},a_{n})}{f(2^{n+m})}\Big)^{\gamma p}\Big]^{\frac{1}{p}}\Big)^{\frac{1}{\gamma}} \\
&\le \Big(\sum_{m=-\infty}^{\infty} \Big(\frac{\min(1,2^m)}{\underline{f}(2^m)}\Big)^{\gamma}\Big[\sum_{n=-\infty}^{\infty}\Big(\frac{J(2^{n},a_{n})}{f(2^{n})}\Big)^{q}\Big]^{\frac{1}{p}}\Big)^{\frac{1}{\gamma}} \\
&= \Big(\sum_{m=-\infty}^{\infty} \Big[\min\Big(1,\frac{1}{2^{-m}}\Big)\bar{f}(2^{-m})\Big]^{\gamma}\Big)^{\frac{1}{\gamma}}\Big[\sum_{n=-\infty}^{\infty}\Big(\frac{J(2^{n},a_{n})}{f(2^{n})}\Big)^{q}\Big]^{\frac{1}{q}}.
\end{aligned}$$

We applied (i) and (ii) of Theorem 2.2.1. The convergence of the integral in the last line follows from Lemma 2.2.4. Taking the infimum over all appropriate representations leads to

$$\|a|A_{f,q;K}\| \le C\|a|A_{f,q;J}\|.$$

Now we show the converse. Assume $a \in A_{f,q;K}$. From Lemma 2.3.1 and property (v) of Theorem 2.2.1 follows that

$$K(t,a) \le c\bar{f}(t)\|a|A_{f,q;K}\| \to 0, \qquad t \to 0$$

and

$$\frac{K(t,a)}{t} \le c\frac{\bar{f}(t)}{t}\|a|A_{f,q;K}\| \to 0, \qquad t \to \infty.$$

Therefore we can apply Lemma 2.1.3 to get a representation $a = \sum_m a_m$ such that $J(2^m,a_m) \le cK(2^m,a)$. It follows

$$\Big(\sum_{m=-\infty}^{\infty} \frac{J(2^m,a_m)^q}{f(2^m)^q}\Big)^{\frac{1}{q}} \le c\Big(\sum_{m=-\infty}^{\infty} \frac{K(2^m,a)^q}{f(2^m)^q}\Big)^{\frac{1}{q}}.$$

This gives the estimate $\|a|A_{f,q;J}\| \leq C\|a|A_{f,q;K}\|$. □

From now on we will omit K and J in the notations.

Theorem 2.3.3 (Interpolation Theorem)**.** *Let* (A_0, A_1) *and* (B_0, B_1) *be compatible couples of quasi-Banach spaces,* $\varrho \in B_{\mathcal{K}}$*, and* $0 < q \leq \infty$*. Then*

(i) $A_0 \cap A_1 \hookrightarrow A_{\varrho,q} \hookrightarrow A_0 + A_1$,

(ii) $A_{\varrho,q}$ *is a quasi-Banach space, and,*

(iii) *if* T *is a bounded linear operator from* A_i *to* B_i *with norm* M_i*,* $i = 0, 1$*, then* T *is bounded from* $A_{\varrho,q}$ *to* $B_{\varrho,q}$ *with norm* M *and*

$$M \leq M_0 \bar{\varrho}\Big(\frac{M_1}{M_0}\Big).$$

Remark 2.3.3. If we put $\varrho(t) = t^\theta$, we have proved Theorem 2.1.1.

Proof. As in [13, p. 295], the proof follows the classical case [3, pp. 47, 64].

(i) For $a \in A_0 \cap A_1$, we have

$$K(t,a) \leq \|a|A_0\| \qquad \text{and} \qquad K(t,a) \leq t\|a|A_1\|,$$

which implies $K(t,a) \leq \min(1,t)\|a|A_0 \cap A_1\|$. It follows that

$$\begin{aligned}
\|a|A_{\varrho,q}\| &= \Big(\int_0^\infty \Big(\frac{K(t,a)}{\varrho(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\leq \Big(\int_0^\infty \Big(\frac{\min(1,t)}{\varrho(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \|a|A_0 \cap A_1\| \\
&\leq c\Big(\int_0^\infty \Big(\min(1,1/t)\bar{\varrho}(t)\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \|a|A_0 \cap A_1\| \\
&= C\|a|A_0 \cap A_1\|.
\end{aligned}$$

Putting $s = 1$ in Lemma 2.3.1 (i) leads to

$$\|a|A_0 + A_1\| \leq c\varrho(1)\|a|A_{\varrho,q}\|.$$

(ii) Because $K(t,a)$ is a equivalent quasi-norm on $A_0 + A_1$ (see Lemma 2.1.2) it only remains to prove the quasi-triangle inequality. Let $a, b \in A_{\varrho,q}$. Then

$$\begin{aligned}\Big(\int_0^\infty \Big(\frac{K(t,a+b)}{\varrho(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} &\le c\Big(\int_0^\infty \Big(\frac{K(t,a)}{\varrho(t)} + \frac{K(t,b)}{\varrho(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\ &\le c\Big(\int_0^\infty \Big(\frac{K(t,a)}{\varrho(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} + \Big(\int_0^\infty \Big(\frac{K(t,b)}{\varrho(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}.\end{aligned}$$

To prove the completeness we use Lemma 1.2.2. Take a sequence $(a_j)_j \subseteq A_{\varrho,q}$ with $\sum_{j=1}^\infty \|a_j|A_{j,q}\|^\gamma < \infty$, where $(2c)^\gamma = 2$ (c is the constant in the quasi-triangle inequality) and $\gamma < q$. Because of $A_{\varrho,q} \hookrightarrow A_0 + A_1$ we have $\sum_{j=1}^\infty \|a_j|A_0 + A_1\|^\gamma < \infty$. Since $A_0 + A_1$ is complete (see Lemma 2.1.1) there is an element $a \in A_0 + A_1$ and $\|a - a_j|A_0 + A_1\| \to 0$.

Then, with the help of Lemma 1.2.2 and Minkowski's inequality,

$$\begin{aligned}\|a|A_{\varrho,q}\| &\sim \Big(\int_0^\infty \Big[\varrho(t)^{-q} K\big(t, \sum_j a_j\big)\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\ &\le \Big(\int_0^\infty \Big[\sum_{j=1}^\infty \Big(\frac{K(t,a_j)}{\varrho(t)}\Big)^\gamma\Big]^{\frac{q}{\gamma}} \frac{dt}{t}\Big)^{\frac{\gamma}{q}\frac{1}{\gamma}} \\ &\le \Big(\sum_{j=1}^\infty \Big[\int_0^\infty \Big(\frac{K(t,a_j)}{\varrho(t)}\Big)^q \frac{dt}{t}\Big]^{\frac{\gamma}{q}}\Big)^{\frac{1}{\gamma}} \\ &= \Big(\sum_{j=1}^\infty \|a_j|A_{\varrho,q}\|^\gamma\Big)^{\frac{1}{\gamma}} < \infty\end{aligned}$$

(iii) It holds

$$\begin{aligned}K(t, Ta; B_0, B_1) &\le \inf_{a=a_0+a_1} \big(\|Ta_0|B_0\| + t\|Ta_1|B_1\|\big) \\ &\le \inf_{a=a_0+a_1} \big(M_0\|a_0|A_0\| + tM_1\|a_1|A_1\|\big).\end{aligned}$$

This gives

$$K(t, Ta; B_0, B_1) \le M_0 K\Big(\frac{M_1}{M_0}t, a; A_0, A_1\Big),$$

and we get

$$\begin{aligned}
\|Ta|B_{\varrho,q}\| &\sim \left(\int_0^\infty \left\{\frac{K(t,Ta;B_0,B_1)}{\varrho(t)}\right\}^q \frac{dt}{t}\right)^{\frac{1}{q}}\\
&\le M_0\left(\int_0^\infty \left\{\frac{K(tM_1/M_0,a;A_0,A_1)}{\varrho(t)}\right\}^q \frac{dt}{t}\right)^{\frac{1}{q}}\\
&= M_0\left(\int_0^\infty \left\{\frac{K(t,a;A_0,A_1)}{\varrho(tM_0/M_1)}\right\}^q \frac{dt}{t}\right)^{\frac{1}{q}}\\
&\le M_0\bar{\varrho}\Big(\frac{M_1}{M_0}\Big)\|a|A_{\varrho,q}\|.
\end{aligned}$$

□

Theorem 2.3.4 (The Reiteration Theorem)**.** *Let (A_0, A_1) a compatible couple of quasi-Banach spaces. Let $0 < q_0, q_1 \le \infty$ and $\varrho_0, \varrho_1 \in FP$ such that $\dfrac{\varrho_1}{\varrho_0}$ is equivalent to a continuously differentiable function τ that satisfies*

$$0 < \delta \le \left|\frac{t\tau'(t)}{\tau(t)}\right| \le b < 1$$

for all $t > 0$ and some δ and b.

Let $\varphi \in FP$ and put

$$\varrho(t) = \varrho_0(t)\varphi\Big(\frac{\varrho_1(t)}{\varrho_0(t)}\Big), \qquad t > 0.$$

Then $\varrho \in FP$ and for $0 < q \le \infty$ holds

$$\big(A_{\varrho_0,q_0}, A_{\varrho_1,q_1}\big)_{\varphi,q} = A_{\varrho,q}$$

with equivalent norms.

Proof. The proof follows the Banach space case in [13, pp. 296 - 299].

At first we assume that

$$0 < \delta \le \frac{t\tau'(t)}{\tau(t)} \le b < 1.$$

Then $\tau \in B_\Psi$. Take $f_0, f_1, g \in B_\Psi$ which are equivalent to $\varrho_0, \varrho_1, \varphi$, respectively and put $\tilde{f}(t) = f_0(t)g\Big(\frac{f_1(t)}{f_0(t)}\Big)$ and $f(t) = f_0(t)g\big(\tau(t)\big)$. Since g is monotonically increasing, it follows that $f \sim \tilde{f} \sim \varrho$. Next we show $\tilde{f} \in B_\Psi$, which implies that $f, \varrho \in FP$. Putting $\mu_i = \inf \frac{tf_i'(t)}{f_i(t)}$, $\mu = \inf \frac{tg'(t)}{g(t)}$, $\nu_i = \sup \frac{tf_i'(t)}{f_i(t)}$, and $\nu = \sup \frac{tg'(t)}{g(t)}$,

$i = 0, 1$, we find

$$\frac{t\tilde{f}'(t)}{\tilde{f}(t)} = \frac{tf_0'(t)}{f_0(t)}\left(1 - \frac{\frac{f_1(t)}{f_0(t)} g'\Big(\frac{f_1(t)}{f_0(t)}\Big)}{g\Big(\frac{f_1(t)}{f_0(t)}\Big)}\right) + \frac{tf_1'(t)}{f_1(t)} \cdot \frac{\frac{f_1(t)}{f_0(t)} g'\Big(\frac{f_1(t)}{f_0(t)}\Big)}{g\Big(\frac{f_1(t)}{f_0(t)}\Big)}.$$

Using $\frac{tf_0'(t)}{f_0(t)} < 1$ we get

$$\begin{aligned}\frac{t\tilde{f}'(t)}{\tilde{f}(t)} &< 1 + \frac{\frac{f_1(t)}{f_0(t)} g'\Big(\frac{f_1(t)}{f_0(t)}\Big)}{g\Big(\frac{f_1(t)}{f_0(t)}\Big)}\Big(\frac{tf_1'(t)}{f_1(t)} - 1\Big)\\ &\leq 1 + \mu(\nu_1 - 1) < 1.\end{aligned}$$

Furthermore

$$\frac{t\tilde{f}'(t)}{\tilde{f}(t)} \geq \mu_0(1-\nu) + \mu_1\mu > 0.$$

Let us draw a few conclusions from $\frac{t\tau'(t)}{\tau(t)} \geq \delta > 0$, which we will need in the proof. It holds

$$\left(\frac{\tau(t)}{t^\delta}\right)' = \Big(\frac{t\tau'(t)}{\tau(t)} + \delta\Big)\frac{\tau(t)}{t^{\delta+1}} \geq 2\delta\frac{\tau(t)}{t^{\delta+1}} > 0,$$

which implies that $\dfrac{\tau(t)}{t^\delta}$ is monotonically increasing. For all $0 < t_1 \leq t_2$ then holds

$$\tau(t_1) \leq \tau(t_2)\Big(\frac{t_1}{t_2}\Big)^\delta$$

and

$$\tau(t_2) \geq \tau(t_1)\Big(\frac{t_2}{t_1}\Big)^\delta.$$

This implies

$$\lim_{t\to 0}\tau(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty}\tau(t) = \infty.$$

Because τ is strictly increasing and continuous, it follows that it has an inverse function η for which holds

$$\frac{1}{b} \leq \frac{s\eta'(s)}{\eta(s)} = \left(\frac{\eta(s)\tau'\big(\eta(s)\big)}{\tau\big(\eta(s)\big)}\right)^{-1} \leq \frac{1}{\delta}.$$

This will be used later on for substitution in integrals.

Now we will show that $\big(A_{f_0,q_0}, A_{f_1,q_1}\big)_{g,q} \hookrightarrow A_{f,q}$. Put $X_i = A_{f_i,q_i}$, $i = 0, 1$. Let $a \in (X_0, X_1)_{g,q}$. Let $a_i \in X_i$, $i = 0, 1$, such that $a = a_0 + a_1$. By Lemma 2.3.1 (i) we get, using the equivalence $\frac{f_1}{f_0} \sim \tau$,

$$\begin{aligned} K(t,a;A_0,A_1) &\le K(t,a_0;A_0,A_1) + K(t,a_1;A_0,A_1) \\ &\le c(f_0(t)\|a_0|X_0\| + f_1(t)\|a_1|X_1\|) \\ &\le cf_0(t)(\|a_0|X_0\| + \tau(t)\|a_1|X_1\|). \end{aligned}$$

It follows that $K(t,a;A_0,A_1) \le cf_0(t)K(\tau(t),a;X_0,X_1)$. If we make the substitution $t = \eta(s)$ and use the observations made above, we get

$$\begin{aligned} \|a|(A_0,A_1)_{f,q}\|^q &\sim \int_0^\infty \left[\frac{K(t,a;A_0,A_1)}{f_0(t)g\big(\tau(t)\big)}\right]^q \frac{dt}{t} \\ &\le c\int_0^\infty \left[\frac{K(\tau(t),a;X_0,X_1)}{g\big(\tau(t)\big)}\right]^q \frac{dt}{t} \\ &= c\int_0^\infty \left[\frac{K(s,a;X_0,X_1)}{g(s)}\right]^q \frac{s\eta'(s)}{\eta(s)}\,\frac{dt}{t} \\ &\le c\int_0^\infty \left[\frac{K(s,a;X_0,X_1)}{g(s)}\right]^q \frac{dt}{t} \\ &= c\|a|\big(A_{f_0,q_0},A_{f_1,q_1}\big)_{g,q}\|^q. \end{aligned}$$

For the converse embedding, Gustavsson uses the substitution method in integrals of Banach space valued functions. Because we want to avoid integration of quasi-Banach space valued functions, we transferred the proof of the quasi-Banach case found in [3, pp. 67-68] to general function parameters.

Take $a \in A_{f,q}$ and let X_i be as above. We will show $\|a|X_{g,q}\| \le c\|a|A_{f,q}\|$. As above, η denotes the inverse function of τ. Then, substituting $t = \tau(s)$, using $\frac{s\tau'(s)}{\tau(s)} < 1$, and changing s back to t, we get

$$\begin{aligned} \|a|X_{g,q}\| &\sim \left(\int_0^\infty \left[\frac{K(t,a;X_0,X_1)}{g(t)}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left[\frac{f_0\big(\eta(t)\big)}{f\big(\eta(t)\big)}K(t,a;X_0,X_1)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\le c\left(\int_0^\infty \left[\frac{f_0(t)}{f(t)}K\big(\tau(t),a;X_0,X_1\big)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\le c\left(\sum_{m=-\infty}^{\infty} \left[\frac{f_0(2^m)}{f(2^m)}K\big(\tau(2^m),a;X_0,X_1\big)\right]^q\right)^{\frac{1}{q}}. \end{aligned}$$

For the last inequality we used Theorem 2.3.1.

We choose a representation $a = \sum\limits_{n=-\infty}^{\infty} u_n$ with $u_n \in A_0 \cap A_1$, such that

$$\sum_{n=-\infty}^{\infty} f(2^n)^{-q} J(2^n, u_n; A_0, A_1) < \infty.$$

Let C be the constant in the quasi-triangle inequality of $A_0 + A_1$, and define γ by $(2C)^\gamma = 2$. If C was chosen large enough, we have $\gamma < q$. From Lemma 1.2.2 then follows for every $m \in \mathbb{Z}$

$$K(t, a; X_0, X_1) \le c\Big(\sum_n K(t, u_{m-n}; X_0, X_1)^\gamma\Big)^{\frac{1}{\gamma}}. \tag{2.3.2}$$

Lemma 2.3.1 (ii) gives

$$J\big(\tau(2^n), u_n; X_0, X_1\big) \le c \max\Big\{\|u_n|X_0\|, \frac{f_1(2^n)}{f_0(2^n)}\|u_n|X_1\|\Big\} \le c\frac{J(2^n, u_n; A_0, A_1)}{f_0(2^n)}.$$

Consequently, applying Lemma 2.1.2, we get

$$\begin{aligned} K\big(\tau(s^m), u_{m-n}; X_0, X_1\big) &\le \min\Big\{1, \frac{\tau(2^m)}{\tau(2^{m-n})}\Big\} J\big(\tau(2^{m-n}), u_{m-n}; X_0, X_1\big) \\ &\le c \min\Big\{1, \frac{\tau(2^m)}{\tau(2^{m-n})}\Big\} \frac{J(2^{m-n}, u_{m-n}; A_0, A_1)}{f_0(2^{m-n})}. \end{aligned} \tag{2.3.3}$$

Combining (2.3.2) and (2.3.3) we see

$$\|a|X_{g,q}\| \le c\bigg(\sum_m f(2^m)^{-q}\Big\{\sum_n f_0(2^m)^\gamma K\big(\tau(2^m), u_{m-n}; X_0, X_1\big)^\gamma\Big\}^{\frac{q}{\gamma}}\bigg)^{\frac{1}{q}}$$

$$\le c\bigg(\sum_m f(2^m)^{-q}\Big\{\sum_n \frac{f_0(2^m)^\gamma}{f_0(2^{m-n})^\gamma} \min\Big\{1, \frac{\tau(2^m)}{\tau(2^{m-n})}\Big\}^\gamma J(2^{m-n}, u_{m-n}; A_0, A_1)^\gamma\Big\}^{\frac{q}{\gamma}}\bigg)^{\frac{1}{q}}.$$

Using Theorem 2.2.1 (ii) we can seperate the variables m and n. In fact, it holds

$$\frac{f_0(2^m)}{f_0(2^{m-n})} \le \frac{f_0(2^m)}{\underline{f}_0(2^{-n}) f(2^m)} = \bar{f}_0(2^n) \quad \text{and} \quad \frac{\tau(2^m)}{\tau(2^{m-n})} \le \bar{\tau}(2^n).$$

Then, since $\frac{q}{\gamma} > 1$, we can apply Minkowski's inequality and get

$$\|a|X_{g,q}\| \le c\Big(\sum_n \bar{f}_0(2^n)^\gamma \min\{1, \bar{\tau}(2^n)\}^\gamma \Big[\sum_m f(2^m)^{-q} J(2^{m-n}, u_{m-n}; A_0, A_1)^q\Big]^{\frac{\gamma}{q}}\Big)^{\frac{1}{\gamma}}.$$

Finally, by replacing m by $m+n$ in the last sum and by

$$f(2^{m+n}) \ge \underline{f}_0(2^n)\underline{g}\big(\underline{\tau}(2^n)\big)\, f_0(2^m) g\big(\tau(2^n)\big)$$

we get

$$\|a|X_{g,q}\| \le c\Bigg(\sum_n \frac{\bar{f}_0(2^n)^\gamma}{\underline{f}_0(2^n)^\gamma} \min\{1, \bar{\tau}(2^n)\}^\gamma \bar{g}\big(\bar{\tau}(2^{-n})\big)^\gamma\Bigg)^{\frac{1}{\gamma}} \Bigg(\sum_m \frac{J(2^m, u_m; A_0, A_1)}{f(2^m)^q}\Bigg)^{\frac{1}{q}}.$$

Now, taking the infimum, we obtain $A_{f,q} \hookrightarrow \big(A_{f_0,q_0}, A_{f_1,q_1}\big)_{g,q}$.

Next we consider the case that $-b \le \dfrac{t\tau'(t)}{\tau(t)} \le -\delta < 0$. We have $\dfrac{t\varrho_1(1/t)}{t\varrho_0(1/t)} \sim \tau(1/t)$ and

$$b \ge \frac{t[\tau(1/t)]'}{\tau(1/t)} = -\frac{1/t\,\tau'(1/t)}{\tau(1/t)} \ge \delta > 0.$$

So, we can apply the first case and Lemma 2.3.1 (iii) leads to

$$\begin{aligned}
\big((A_0,A_1)_{\varrho_0,q_0}, (A_0,A_1)_{\varrho_1,q_1}\big)_{\varphi,q} &= \big((A_1,A_0)_{t\varrho_0(1/t),q_0}, (A_1,A_0)_{t\varrho_1(1/t),q_1}\big)_{\varphi,q} \\
&= (A_1,A_0)_{t\varrho_0(1/t)\varphi\left(\frac{\varrho_1(1/t)}{\varrho_0(1/t)}\right),q} \\
&= (A_0,A_1)_{\varrho_0(t)\varphi\left(\frac{\varrho_1(t)}{\varrho_0(t)}\right),q}.
\end{aligned}$$

□

Remark 2.3.4. To formulate the condition on the functions ϱ_0 and ϱ_1 we replaced $\tau(t) = \frac{\varrho_1(t)}{\varrho_0(t)}$ by $\tau \sim \frac{\varrho_1}{\varrho_0}$. We have done this, because later on we want to apply the theorem to function parameters $\varrho_i(t) = t^{\theta_i} b_i(t)$, $i = 0, 1$, with slowly varying b_i, and these are not contained in B_Ψ and do not even have to be differentiable. See Corollary 2.4.1.

If we put $\varrho_i(t) = t^{\theta_i}$ and $\varphi(t) = t^\eta$ in the above theorem, where $\theta = (1-\eta)\theta_0 + \eta\theta_1$, we get $\varrho(t) = t^\theta$ and receive the well known result stated in Theorem 2.1.4.

Lemma 2.3.2. *Let $\varrho \in FP$ and let (A_0, A_1) be a compatible couple of quasi-Banach* *spaces. Let* $0 < q \le \tilde{q} \le \infty$. *Then*

$$A_{\varrho,q} \hookrightarrow A_{\varrho,\tilde{q}}.$$

Proof. If $\tilde{q} = \infty$, the inequality of the quasi-norms follows directly from Lemma 2.3.1. Let now $0 < q \le \tilde{q} < \infty$. Using Lemma 2.3.1 we get

$$\begin{aligned}\|a|A_{\varrho,\tilde{q}}\| &= \left(\int_0^\infty \big(\varrho(t)^{-1}K(t,a)\big)^q \big(\varrho(t)^{-1}K(t,a)\big)^{\tilde{q}-q}\,\frac{dt}{t}\right)^{\frac{1}{\tilde{q}}}\\ &\le c\|a|A_{\varrho,q}\|^{\frac{\tilde{q}-q}{\tilde{q}}}\,\|a|A_{\varrho,q}\|^{\frac{q}{\tilde{q}}}\\ &= c\|a|A_{\varrho,q}\|\end{aligned}$$

□

Remark 2.3.5. Choosing $\varrho(t) = t^\theta$, we get the assertion in Theorem 2.1.3 (ii).

2.4 Interpolation with slowly varying parameters

Let $b \in SV$ and $\theta \in (0,1)$. We have seen in Lemma 2.2.1 that $t^\theta b(t) \in FP$. So, we can use $t^\theta b(t)$ as a function parameter and we have

$$\begin{aligned}\|a|A_{t^\theta b(t),q}\| &= \left(\sum_{m=-\infty}^{\infty} 2^{-m\theta q} b(2^m)^{-q} K(2^m,a)^q\right)^{\frac{1}{q}}\\ &\sim \left(\int_0^\infty t^{-\theta q} b(t)^{-q} K(t,a)^q\,\frac{dt}{t}\right)^{\frac{1}{q}}.\end{aligned}$$

In the following theorem we collect some properties of interpolation spaces with parameter functions $t^\theta b(t)$.

Theorem 2.4.1. *Let* $A = (A_0, A_1)$ *a compatible couple of quasi-Banach spaces. Let* $b \in SV$, $\theta \in (0,1)$, *and* $0 < q \le \infty$.

(i) *We have* $(A_0, A_1)_{t^\theta b(t),q} = (A_1, A_0)_{t^{1-\theta} b(t^{-1}),q}$.

(ii) *It holds* $K(s,a) \le c\, s^\theta\, b(s)\, \|a|A_{t^\theta b(t),q}\|$ *for all* $s > 0$.

(iii) *Let* $A_0 \hookrightarrow A_1$. *Let* $\tilde{\theta} \in (0,1)$ *such that* $\theta < \tilde{\theta}$, *and let* $0 < q, \tilde{q} \le \infty$. *Then*

$$A_{t^\theta b(t),q} \hookrightarrow A_{t^{\tilde{\theta}} b(t),\tilde{q}}.$$

Remark 2.4.1. If we put $b(t) \equiv 1$ in part (iii) of the theorem, we get the monotonicity of classical real interpolation spaces stated in Theorem 2.1.3 (iii).

Proof. (i) and (ii) follow immediately from Lemma 2.3.1.

We prove (iii). The proof is carried over from the classical case, see [21, pp. 25-27] and [3, pp. 46-47]. From Lemma 2.1.2 and the definition of the K-functional it follows from $A_0 \hookrightarrow A_1$ that $K(t,a) = t\|a|A_1\|$ for all $t \in (0,1)$. Let $\theta < \tilde{\theta}$ and let $0 < \tilde{q} < \infty$. By Theorem 2.3.3, $A_{t^\theta b(t),\infty}$ is an intermediate space. We will now need the fact that is is continuously embedded in A_1. We can write

$$\begin{aligned}
\|a|A_{t^{\tilde{\theta}}b(t),\tilde{q}}\|^{\tilde{q}} &= \int_0^1 t^{-\tilde{\theta}\tilde{q}} b(t)^{-\tilde{q}} K(t,a)^{\tilde{q}} \frac{dt}{t} + \int_1^\infty t^{-\tilde{\theta}\tilde{q}} b(t)^{-\tilde{q}} K(t,a)^{\tilde{q}} \frac{dt}{t} \\
&= \|a|A_1\|^{\tilde{q}} \int_0^1 t^{(1-\tilde{\theta})\tilde{q}} b(t)^{-\tilde{q}} \frac{dt}{t} + \int_1^\infty t^{(\theta-\tilde{\theta})\tilde{q}} t^{-\theta\tilde{q}} b(t)^{-\tilde{q}} K(t,a)^{\tilde{q}} \frac{dt}{t} \\
&\le c\|a|A_{t^\theta b(t),\infty}\|^{\tilde{q}} \int_0^1 \left(\frac{t^{(1-\tilde{\theta}-\varepsilon)}}{t^{-\varepsilon} b(t)}\right)^{\tilde{q}} \frac{dt}{t} + \|a|A_{t^\theta b(t),\infty}\|^{\tilde{q}} \int_1^\infty t^{(\theta-\tilde{\theta})\tilde{q}} \frac{dt}{t} \\
&\le c\|a|A_{t^\theta b(t),\infty}\|^{\tilde{q}},
\end{aligned}$$

where we used that $t^{-\varepsilon}b(t)$ is equivalent to a non-increasing function for each $\varepsilon \in (0,1)$. Then we chose $\varepsilon < 1 - \tilde{\theta}$ to get a finite integral.

Now we choose q with $0 < q \le \infty$. Then from Lemma 2.3.2 follows

$$A_{t^\theta b(t),q} \hookrightarrow A_{t^\theta b(t),\infty} \hookrightarrow A_{t^{\tilde{\theta}} b(t),\tilde{q}} \hookrightarrow A_{t^{\tilde{\theta}} b(t),\infty}.$$

□

In the following corollary we formulate the reiteration theorem for slowly varying parameters. It has been stated in [11, p. 91].

Corollary 2.4.1. *Let (A_0, A_1) a compatible couple of quasi-Banach spaces. For $i = 0,1$ suppose $0 < q_i \le \infty$, $b_i \in SV$, and $\theta_i \in (0,1)$ with $\theta_0 \ne \theta_1$.*

Take $d \in SV$, $\eta \in (0,1)$, and put

$$\theta = (1-\eta)\theta_0 + \eta\theta_1 \quad \textit{and} \quad b(t) = b_0(t)^{1-\eta} b_1(t)^\eta d\Big(t^{\theta_1-\theta_0}\frac{b_1(t)}{b_0(t)}\Big).$$

Then b is slowly varying and for every $0 < q \le \infty$ holds

$$\Big(A_{t^{\theta_0}b_0(t),q_0}, A_{t^{\theta_1}b_1(t),q_1}\Big)_{t^\eta d(t),q} = A_{t^\theta b(t),q}.$$

Proof. From Theorem 1.4.1 (iv) follows that $b \in SV$. To prove the corollary, we will apply Theorem 2.3.4 with $\varrho_i(t) = t^{\theta_i} b_i(t)$, $i = 0, 1$, and $\varphi(t) = t^{\eta} d(t)$. To this end, we have to show that $\frac{\varrho_1(t)}{\varrho_0(t)} = t^{\theta_1 - \theta_0} \frac{b_1(t)}{b_0(t)}$ satisfies the condition in Theorem 2.3.4. If $\theta_0 < \theta_1$, it follows from Lemmas 2.2.1 and 2.2.5 that $t^{\theta_1 - \theta_0} \frac{b_1(t)}{b_0(t)}$ is equivalent to a function $\tau \in B_\Psi$. Hence, the assumption for Theorem 2.3.4 is fulfilled. On the other hand, if $\theta_1 < \theta_0$, we have that $t^{\theta_1 - \theta_0} \frac{b_1(t)}{b_0(t)}$ is equivalent to a function $\frac{1}{\tau}$ with $\tau \in B_\Psi$. Observing that

$$\frac{t\big(\frac{1}{\tau(t)}\big)'}{\frac{1}{\tau(t)}} = -\frac{t\tau'(t)}{\tau(t)}$$

we see that we can apply Theorem 2.3.4 in this case, too. □

Proposition 2.4.1. *Let $0 < r < q \leq \infty$. Let $b \in SV$, $\theta \in (0,1)$, $\frac{1}{p} = \frac{1-\theta}{r}$, and $\tilde{b}(t) = b(t^{1/r})^{-1}$. Then*

$$(L_r, L_\infty)_{t^\theta b(t), q} = L_{p,q;\tilde{b}}.$$

Proof. In Theorem 1.4.1 (i) we showed that $\tilde{b}$ is slowly varying. Then, utilizing Example 2.1.1, we get

$$\begin{aligned}
&\|f|\big(L_r(\Omega), L_\infty(\Omega)\big)_{t^\theta b(t), q}\| \\
&\sim \left[\int_0^\infty t^{-\theta q} b(t)^{-q} \bigg(\int_0^{t^r} s f^*(s)\,\frac{ds}{s}\bigg)^{\frac{q}{r}} \frac{dt}{t}\right]^{\frac{1}{q}} \\
&= \left[\int_0^\infty t^{-\theta q} b(t)^{-q} \bigg(\int_0^{1} s t^r f^*(s t^r)^r\,\frac{ds}{s}\bigg)^{\frac{q}{r}} \frac{dt}{t}\right]^{\frac{1}{q}} \\
&= \left[\int_0^\infty \bigg(\int_0^{1} s t^{(1-\theta)r} b(t)^{-r} f^*(s t^r)^r\,\frac{ds}{s}\bigg)^{\frac{q}{r}} \frac{dt}{t}\right]^{\frac{r}{q}\frac{1}{r}} \\
&\leq \left\{\int_0^1 \left[\int_0^\infty s^{\frac{q}{r}} t^{(1-\theta)q} b(t)^{-q} f^*(s t^r)^q\,\frac{dt}{t}\right]^{\frac{r}{q}} \frac{ds}{s}\right\}^{\frac{1}{r}} \\
&= c\bigg\{\int_0^1 \left[s^{-\frac{(1-\theta)q}{r}} \int_0^\infty t^{\frac{(1-\theta)q}{r}} b\big(s^{-\frac{1}{r}} t^{\frac{1}{r}}\big)^{-q} f^*(t)^q\,\frac{dt}{t}\right]^{\frac{r}{q}} ds\bigg\}^{\frac{1}{r}} \\
&\leq c\bigg(\int_0^1 s^{-(1-\theta)-\frac{\varepsilon}{q}}\,ds\bigg)^{\frac{1}{r}} \bigg(\int_0^\infty t^{\frac{(1-\theta)q}{r}} b\big(t^{\frac{1}{r}}\big)^{-q} f^*(t)^q\,\frac{dt}{t}\bigg)^{\frac{1}{q}} \\
&= c\bigg(\int_0^1 \frac{ds}{s^{1-\theta+\frac{\varepsilon}{q}}}\bigg)^{\frac{1}{r}} \bigg(\int_0^\infty t^{\frac{q}{p}} \tilde{b}(t)^q f^*(t)^q\,\frac{dt}{t}\bigg)^{\frac{1}{q}} \\
&= c\|f|L_{p,q;\tilde{b}}(\Omega)\|,
\end{aligned}$$

where we applied Minkowski's inequality and used the property of slowly varying functions stated in Theorem 1.4.1 (ii).

Conversely, starting with the third line of the previous estimate and using that f^* is non-increasing,

$$\begin{aligned}\|f|\big(L_r(\Omega), L_\infty(\Omega)\big)_{t^\theta b(t),q}\| &\geq \left(\int_0^\infty t^{(1-\theta)q} b(t)^{-q} f^*(t^r)^q \,\frac{dt}{t}\right)^{\frac{1}{q}} \\ &= c\left(\int_0^\infty t^{\frac{(1-\theta)q}{r}} b\big(t^{\frac{1}{r}}\big)^{-q} f^*(t)^q \,\frac{dt}{t}\right)^{\frac{1}{q}} \\ &= c\|f|L_{p,q;\tilde{b}}(\Omega)\|.\end{aligned}$$

□

On the basis of Proposition 2.4.1 we can prove the following theorem. This has been done in [11, p.100].

Theorem 2.4.2. *Let $b_i \in SV$, $0 < p_i, q_i, q \leq \infty$, $i = 0, 1$, such that $p_0 \neq p_1$ and let $\eta \in (0,1)$ and $d \in SV$.*

Put $\dfrac{1}{p} = \dfrac{1-\eta}{p_0} + \dfrac{\eta}{p_1}$ and

$$b(t) = b_0(t)^{1-\eta} b_1(t)^\eta d\Big(t^{\frac{1}{p_0}-\frac{1}{p_1}} \frac{b_0(t)}{b_1(t)}\Big)^{-1}.$$

Then

$$\big(L_{p_0,q_0;b_0}(\Omega), L_{p_1,q_1;b_1}(\Omega)\big)_{t^\eta d(t),q} = L_{p,q;b}(\Omega).$$

Proof. Let $r < \min(p_0, p_1)$ and let θ_i be defined by $\frac{1}{p_i} = \frac{1-\theta_i}{r}$, $i = 0, 1$. Applying Proposition 2.4.1 we get (we omit (Ω) temporarily)

$$\big(L_{p_0,q_0;b_0}, L_{p_1,q_1;b_1}\big)_{t^\eta d(t),q} = \Big((L_r, L_\infty)_{t^{\theta_0} b_0(t^r)^{-1},q_0}, (L_r, L_\infty)_{t^{\theta_1} b_1(t^r)^{-1},q_1}\Big)_{t^\eta d(t),q}.$$

Now, put $\theta = (1-\eta)\theta_0 + \eta\theta_1$ and

$$\tilde{b}(t) = \frac{1}{b_0(t^r)^{1-\eta}} \frac{1}{b_1(t^r)^\eta} d\Big(t^{\theta_1-\theta_0} \frac{b_0(t^r)}{b_1(t^r)}\Big).$$

Hence, using the Reiteration Theorem (Corollary 2.4.1),

$$\big(L_{p_0,q_0;b_0}, L_{p_1,q_1;b_1}\big)_{t^\eta d(t),q} = \big(L_r, L_\infty\big)_{t^\theta \tilde{b}(t),q}.$$

Observing that $p = \frac{1-\theta}{r}$ and $b(t) = \tilde{b}\big(t^{\frac{1}{r}}\big)^{-1}$ we apply again Proposition 2.4.1 and the proof is complete. □

2.5 Interpolation with logarithmic parameters

Let

$$\begin{aligned} l_1(t) &:= 1 + |\log t|, \\ l_2(t) &:= 1 + \big|\log\big(1 + |\log t|\big)\big|, \\ l_{i+1}(t) &:= 1 + \big|\log\big(l_i(t)\big)\big|, \quad \text{for } i \in \mathbb{N}. \end{aligned}$$

For $N \in \mathbb{N}$ and $\bar{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ we put

$$\lambda_{\bar{\alpha}}(t) = \prod_{i=1}^{N} l_i(t)^{\alpha_i}.$$

In Example 1.4.1 we saw that $\lambda_{\bar{\alpha}}$ is slowly varying. So we can use it for interpolation as in the previous section. For $0 < q \leq \infty$, we denote the outcoming spaces by

$$A_{\theta,\bar{\alpha},q} = A_{\theta,\alpha_1,\dots,\alpha_N,q} := A_{t^\theta \lambda_{\bar{\alpha}}(t),q}.$$

That means

$$\begin{aligned} \|a|A_{\theta,\alpha_1,\dots,\alpha_N,q}\| &= \left(\sum_{m=-\infty}^{\infty} \left\{ 2^{-m\theta} l_1(2^m)^{-\alpha_1} \cdots l_N(2^m)^{-\alpha_N} K(2^m, a) \right\}^q \right)^{\frac{1}{q}} \\ &\sim \left(\int_0^\infty \left\{ t^{-\theta} l_1(t)^{-\alpha_1} \cdots l_N(t)^{-\alpha_N} K(t, a) \right\}^q \frac{dt}{t} \right)^{\frac{1}{q}} \end{aligned}$$

Lemma 2.5.1. (i) *Let $f, g\colon (0,\infty) \to (0,\infty)$ such that $f \sim g$. Then $l_1 \circ f \sim l_1 \circ g$.*

(ii) *Let $r \in \mathbb{R}$. Then $l_1(t^r) \sim l_1(t)$.*

(iii) *Let $N \in \mathbb{N}$, $\bar{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$, and $r \in \mathbb{R}$. Then $\lambda_{\bar{\alpha}}(t^r) \sim \lambda_{\bar{\alpha}}(t)$.*

Proof. (i) We have $f(t) \leq c_1 g(t)$ and $g(t) \leq c_2 f(t)$ for all $t > 0$. If $f(t) \geq 1$, we can write

$$1 + |\log\big(f(t)\big)| \leq 1 + |\log\big(c_1 g(t)\big)| \leq (1 + |\log c_1|)\Big(1 + |\log\big(g(t)\big)|\Big).$$

If $0 < f(t) < 1$, then

$$1 + |\log\big(f(t)\big)| \leq 1 + |\log\big(\tfrac{1}{c_2} g(t)\big)| \leq (1 + |\log \tfrac{1}{c_2}|)\Big(1 + |\log\big(g(t)\big)|\Big).$$

So, we arrived at

$$1+|\log\bigl(f(t)\bigr)| \le \max(1+|\log c_1|, 1+|\log c_2|)\Bigl(1+|\log\bigl(g(t)\bigr)|\Bigr)$$

for all $t>0$. The same holds if we exchange f and g.

(ii) With $r \in \mathbb{R}$ we have

$$\min(1,|r|)(1+|\log t|) \le 1+|r||\log t| \le \max(1,|r|)(1+|\log t|).$$

(iii) From (i) and (ii) it follows $l_i(t^r) \sim l_i(t)$ for all $i=1,\dots,N$. Consequently, we get (iii). □

The following lemma shows that, if $A_0 \hookrightarrow A_1$, we only have to consider $t>1$ to characterize logarithmic interpolation spaces. This property is mentioned in [5, p. 234].

Lemma 2.5.2. *Let (A_0, A_1) with $A_0 \hookrightarrow A_1$, $\theta \in (0,1)$, and $0<q\le\infty$. Let $N \in \mathbb{N}$ and $\bar{\alpha} = (\alpha_1,\dots,\alpha_N) \in \mathbb{R}^N$. Then*

$$\|a|A_{\theta,\bar{\alpha},q}\| \sim \left(\int_1^\infty t^{-\theta q}\lambda_{\bar{\alpha}}(t)^{-q}K(t,a)^q\,\frac{dt}{t}\right)^{\frac{1}{q}} \sim \left(\sum_{m=1}^\infty t^{-\theta q}\lambda_{\bar{\alpha}}(2^m)^{-q}K(2^m,a)^q\right)^{\frac{1}{q}}.$$

Proof. Because of $A_0 \hookrightarrow A_1$ it holds, if $0<t<1$, that $K(t,a) = t\|a|A_1\|$. Using Lemma 2.5.1 (iii) and Lemma 2.1.2 we can write

$$\begin{aligned}\int_0^1 t^{-\theta q}\lambda_{\bar{\alpha}}(t)^{-q}K(t,a)^q\,\frac{dt}{t} &= \int_0^1 t^{(1-\theta)q}\lambda_{\bar{\alpha}}(t)^{-q}\|a|A_1\|^q\,\frac{dt}{t}\\ &= \frac{\theta}{1-\theta}\int_1^\infty \tau^{-\theta q}\lambda_{\bar{\alpha}}(\tau)^{-q}\|a|A_1\|^q\,\frac{d\tau}{\tau}\\ &\le \frac{\theta}{1-\theta}\int_1^\infty \tau^{-\theta q}\lambda_{\bar{\alpha}}(\tau)^{-q}K(\tau,a)^q\,\frac{d\tau}{\tau}.\end{aligned}$$

Here we made the transformation $t = \tau^{-\frac{\theta}{1-\theta}}$. Now we have

$$\|a|A_{\theta,\bar{\alpha},q}\|^q \le \frac{1}{1-\theta}\int_1^\infty t^{-\theta q}\lambda_{\bar{\alpha}}(t)^{-q}K(t,a)^q\,\frac{dt}{t}.$$

Since the converse inequality is trivial, the result follows. □

Lemma 2.5.3. (i) *For all $\varepsilon > 0$ and $\bar{\alpha} \in \mathbb{R}^N$ it holds*

$$\int_1^\infty \frac{dt}{t^{1+\varepsilon}\lambda_{\bar{\alpha}}(t)} < \infty.$$

(ii) *Let $j \in \{1, \ldots, N\}$ and $\alpha_{j+1}, \ldots, \alpha_N \in \mathbb{R}$. Then*

$$\int_1^\infty \frac{dt}{t\, l_1(t) \cdots l_{j-1}(t)\, l_j(t)^{1+\varepsilon}\, l_{j+1}(t)^{\alpha_{j+1}} \cdots l_N(t)^{\alpha_N}} < \infty,$$

for all $\varepsilon > 0$.

Proof. (i) Because $t^\delta l_i(t)^{\alpha_i} \to \infty$ as $t \to \infty$ for all $\delta > 0$ and $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, N$, we have

$$\int_1^\infty \frac{dt}{t^{1+\varepsilon}\lambda_{\bar{\alpha}}(t)} = \int_1^\infty \frac{1}{t^{1+\varepsilon/2}} \cdot \frac{1}{t^{\varepsilon/2}\lambda_{\bar{\alpha}}(t)}\, dt \le c \int_1^\infty \frac{dt}{t^{1+\varepsilon/2}} < \infty.$$

(ii) Let $\varepsilon > 0$. For $t > 1$ let $f(t) := -\frac{1}{\varepsilon} l_j(t)^{-\varepsilon}$. Then

$$f'(t) = l_j(t)^{-\varepsilon-1} \cdot l_j'(t) = \frac{1}{t\, l_1(t) \cdots l_{j-1}(t)\, l_j(t)^{1+\varepsilon}}.$$

Therefore

$$\int_1^\infty \frac{dt}{t\, l_1(t) \cdots l_{j-1}(t)\, l_j(t)^{1+\varepsilon}} = \frac{1}{\varepsilon l_j(1)^\varepsilon} = \frac{1}{\varepsilon}.$$

Then, as in the proof of (i),

$$\begin{aligned}
&\int_1^\infty \frac{dt}{t\, l_1(t) \cdots l_{j-1}(t)\, l_j(t)^{1+\varepsilon}\, l_{j+1}(t)^{\alpha_{j+1}} \cdots l_N(t)^{\alpha_N}} \\
&= \int_1^\infty \frac{1}{t\, l_1(t) \cdots l_{j-1}(t)\, l_j(t)^{1+\varepsilon/2}} \cdot \frac{1}{l_j(t)^{\varepsilon/2}\, l_{j+1}(t)^{\alpha_{j+1}} \cdots l_N(t)^{\alpha_N}}\, dt \\
&\le c \int_1^\infty \frac{dt}{t\, l_1(t) \cdots l_{j-1}(t)\, l_j(t)^{1+\varepsilon/2}} < \infty.
\end{aligned}$$

□

The next theorem is a generalization of a theorem about Lorentz-Zygmund spaces given in [1, pp.31-32].

Theorem 2.5.1. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $N \in \mathbb{N}$.*

(i) *Let* $0 < \theta < \tilde{\theta} < 1$, $\bar{\alpha}^0, \bar{\alpha}^1 \in \mathbb{R}^N$, *and* $0 < q, \tilde{q} \le \infty$. *Then*

$$A_{\theta,\bar{\alpha}^0,q} \hookrightarrow A_{\tilde{\theta},\bar{\alpha}^1,\tilde{q}}.$$

(ii) *Let* $\theta \in (0,1)$, $\bar{\alpha}^0 = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$, *and* $\bar{\alpha}^1 = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N) \in \mathbb{R}^N$. *If there is a number* $j \in \{1, \ldots, N\}$ *such that either*

$$q \le \tilde{q}, \quad \alpha_k = \tilde{\alpha}_k \quad \text{for } k = 1, \ldots, j-1, \quad \text{and} \quad \alpha_j < \tilde{\alpha}_j \tag{2.5.1}$$

or

$$q > \tilde{q}, \quad \tilde{\alpha}_k - \alpha_k = \frac{1}{\tilde{q}} - \frac{1}{q}, \quad \text{for } k = 1, \ldots, j-1, \quad \text{and} \quad \tilde{\alpha}_j - \alpha_j > \frac{1}{\tilde{q}} - \frac{1}{q}, \tag{2.5.2}$$

then

$$A_{\theta,\bar{\alpha}^0,q} \hookrightarrow A_{\theta,\bar{\alpha}^1,\tilde{q}}.$$

Remark 2.5.1. For a better understanding we want to discuss the result of the theorem for $N = 2$. We have to consider the embedding

$$A_{\theta,\alpha_1,\alpha_2,q} \hookrightarrow A_{\tilde{\theta},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{q}}.$$

Part (i) says that, if $\theta < \tilde{\theta}$, the "distance" of the spaces is so big that the influence of the parameters α_1, α_2, and q is too small to affect the embedding.

In part (ii) the spaces come closer, i.e. we have $\theta = \tilde{\theta}$. Therefore α_1 is the next most dominant parameter. But now, in contrast to (i), the relation between q and $\tilde{q}$ is important.

Suppose that $q \le \tilde{q}$. Then, if $\alpha_1 < \tilde{\alpha}_1$, we get the embedding for any α_2 and $\tilde{\alpha}_2$. If, on the other hand, $\alpha_1 = \tilde{\alpha}_1$, then the finer tuning is done by α_2: the embedding holds if $\alpha_2 < \tilde{\alpha}_2$.

Suppose that $q > \tilde{q}$. Now $\alpha_1 < \tilde{\alpha}_1$ is not sufficient. But if the distance between α_1 and $\tilde{\alpha}_1$ is greater than $\frac{1}{\tilde{q}} - \frac{1}{q}$, the embedding holds for any α_2 and $\tilde{\alpha}_2$. If $\tilde{\alpha}_1 - \alpha_1 = \frac{1}{\tilde{q}} - \frac{1}{q}$ then we need $\tilde{\alpha}_2 - \alpha_2 > \frac{1}{\tilde{q}} - \frac{1}{q}$ to prove the embedding. The reason for that is that in the proof we will need the appearing integrals to be convergent. We showed this convergence in Lemma 2.5.3.

Proof. (i) Let $0 < \theta < \tilde{\theta} < 1$. Then

$$\begin{aligned}
\|a|A_{\tilde{\theta},\bar{\alpha}^1,\tilde{q}}\|^{\tilde{q}} &= \int_1^\infty \{t^{-\tilde{\theta}}\lambda_{\bar{\alpha}^1}(t)^{-1}K(t,a)\}^{\tilde{q}}\,\frac{dt}{t} \\
&= \int_1^\infty \{t^{-\theta}\lambda_{\bar{\alpha}^0}(t)^{-1}K(t,a)\}^{\tilde{q}}\cdot\{t^{\theta-\tilde{\theta}}\lambda_{\bar{\alpha}^0-\bar{\alpha}^1}(t)\}^{\tilde{q}}\,\frac{dt}{t} \\
&\le \|a|A_{\theta,\bar{\alpha}^0,\infty}\|^{\tilde{q}}\cdot\int_1^\infty \frac{dt}{t^{1+(\tilde{\theta}-\theta)\tilde{q}}\lambda_{\bar{\alpha}^1-\bar{\alpha}^0}(t)^{\tilde{q}}} \\
&\le \|a|A_{\theta,\bar{\alpha}^0,\infty}\|^{\tilde{q}}
\end{aligned}$$

according to Lemma 2.5.3 (i). Then the result follows from Lemma 2.3.2.

(ii) Firstly, let $q = \tilde{q}$ and $\bar{\alpha}^0$ and $\bar{\alpha}^1$ as in (2.5.1). Then

$$\begin{aligned}
&\|a|A_{\theta,\bar{\alpha}^1,q}\|^q \\
&= \int_1^\infty \left\{t^{-\theta}\,l_1(t)^{-\tilde{\alpha}_1}\cdots l_N(t)^{-\tilde{\alpha}_N}\,K(t,a)\right\}^q\,\frac{dt}{t} \\
&= \int_1^\infty \left\{t^{-\theta}\,l_1(t)^{-\alpha_1}\cdots l_N(t)^{-\alpha_N}\,K(t,a)\right\}^q\cdot\left\{l_j(t)^{\alpha_j-\tilde{\alpha}_j}\cdots l_N(t)^{\alpha_N-\tilde{\alpha}_N}\right\}^q\frac{dt}{t}.
\end{aligned}$$

Because $\tilde{\alpha}_j - \alpha_j > 0$ the term $l_j(t)^{\alpha_j-\tilde{\alpha}_j}\cdots l_N(t)^{\alpha_N-\tilde{\alpha}_N}$ tends to zero as $t\to\infty$. So the last integral is smaller or equal $c\|a|A_{\theta,\bar{\alpha}^0,q}\|^q$ with $c > 0$.

Then, if $q \le \tilde{q}$, we get with Lemma 2.3.2

$$A_{\theta,\bar{\alpha}^0,q} \hookrightarrow A_{\theta,\bar{\alpha}^1,q} \hookrightarrow A_{\theta,\bar{\alpha}^1,\tilde{q}}.$$

Now, let condition (2.5.2) be true. Then, by applying Hölder's inequality with conjugate exponents $\frac{q}{\tilde{q}}$ and $\frac{q}{q-\tilde{q}}$, we find

$$\begin{aligned}
\|a|A_{\theta,\bar{\alpha}^1,\tilde{q}}\|^{\tilde{q}} &= \int_1^\infty \left\{t^{-\theta}\lambda_{\bar{\alpha}^0}(t)^{-1}K(t,a)\right\}^{\tilde{q}}\cdot\lambda_{\bar{\alpha}^0-\bar{\alpha}^1}(t)^{\tilde{q}}\frac{dt}{t} \\
&\le \left(\int_1^\infty \left\{t^{-\theta}\lambda_{\bar{\alpha}^0}(t)^{-1}K(t,a)\right\}^q\frac{dt}{t}\right)^{\frac{\tilde{q}}{q}}\cdot\left(\int_1^\infty \frac{dt}{t\,\lambda_{\frac{q\tilde{q}}{q-\tilde{q}}(\bar{\alpha}^1-\bar{\alpha}^0)}(t)}\right)^{1-\frac{\tilde{q}}{q}}.
\end{aligned} \tag{2.5.3}$$

Condition (2.5.2) implies

$$\begin{aligned}
&\lambda_{\frac{q\tilde{q}}{q-\tilde{q}}(\bar{\alpha}^1-\bar{\alpha}^0)}(t) \\
&= l_1(t)\cdots l_{j-1}(t)\left(l_j(t)^{\tilde{\alpha}_j-\alpha_j}\,l_{j+1}(t)^{\tilde{\alpha}_{j+1}-\alpha_{j+1}}\cdots l_N^{\tilde{\alpha}_N-\alpha_N}\right)^{\frac{q\tilde{q}}{q-\tilde{q}}},
\end{aligned}$$

and $\frac{q\tilde{q}}{q-\tilde{q}}(\tilde{\alpha}_j - \alpha_j) > 1$. Consequently, the last integral in (2.5.3) is convergent as we showed in Lemma 2.5.3 (ii), and the assertion is proved. $\square$

Remark 2.5.2. If we have $\alpha_i = \tilde{\alpha}_i$ for all $i = 1, \dots, N$ and $q \leq \tilde{q}$ in Theorem 2.5.1 (ii), then

$$A_{\theta,\bar{\alpha}^0,q} \hookrightarrow A_{\theta,\bar{\alpha}^0,\tilde{q}}.$$

This follows from Lemma 2.3.2.

Remark 2.5.3. If the couple (A_0, A_1) is not ordered, we still have

$$A_{\theta,\bar{\alpha}^0,q} \hookrightarrow A_{\theta,\bar{\alpha}^1,q}$$

if $\alpha_i \leq \tilde{\alpha}_i$ for all $i = 1, \dots, N$. This follows directly from the definition of the spaces, because $l_i(t) \geq 1$ for $i = 1, \dots, N$ and all $t > 0$.

Next, we state the Reiteration Theorem for logarithmic parameters. Compare [8, p. 921] and [10, p. 241].

Corollary 2.5.1. *Let (A_0, A_1) a compatible couple of quasi-Banach spaces. For $i = 0, 1$ let $0 < q_i \leq \infty$, $\bar{\alpha}^0, \bar{\alpha}^1 \in \mathbb{R}^N$, and let $\theta_0 \neq \theta_1$. Then, for $\bar{\beta} \in \mathbb{R}^N$, $\eta \in (0, 1)$, and $0 < q \leq \infty$, we have*

$$\left(A_{\theta_0,\bar{\alpha}^0,q_0}, A_{\theta_1,\bar{\alpha}^1,q_1}\right)_{\eta,\bar{\beta},q} = A_{\theta,\bar{\alpha},q},$$

where

$$\theta = (1-\eta)\theta_0 + \eta\theta_1 \qquad \text{and} \qquad \bar{\alpha} = (1-\eta)\bar{\alpha}^0 + \eta\bar{\alpha}^1 + \bar{\beta}.$$

Proof. We apply Corollary 2.4.1 with $b_i(t) = \lambda_{\bar{\alpha}^i}(t)$ and $d(t) = \lambda_{\bar{\beta}}(t)$. Because $\lambda_{\bar{\beta}}\Big(t^{\theta_1-\theta_0}\lambda_{\bar{\alpha}^1-\bar{\alpha}^0}(t)\Big) \sim \lambda_{\bar{\beta}}(t)$, the corollary is proved. $\square$

Now we specify Corollary 2.4.1 choosing $b \in SV$ as $\lambda_{\bar{\alpha}}$.

Corollary 2.5.2. *Let $0 < r < q \leq \infty$. Let $\bar{\alpha} \in \mathbb{R}^N$, $\theta \in (0, 1)$, and $\frac{1}{p} = \frac{1-\theta}{r}$. Then*

$$\big(L_r(\Omega), L_\infty(\Omega)\big)_{\theta,\bar{\alpha},q} = L_{p,-\bar{\alpha},q}(\Omega).$$

Proof. The assertion follows directly from Corollary 2.4.1 by observing that

$$\tilde{b}(t) = \lambda_{\bar{\alpha}}\big(t^{\frac{1}{r}}\big)^{-1} \sim \lambda_{-\bar{\alpha}}(t).$$

$\square$

Finally, we interpolate generalized Lorentz-Zygmund spaces.

Corollary 2.5.3. *Let* $\bar{\alpha}^i \in \mathbb{R}^N$, $0 < p_i, q_i, q \le \infty$, $i = 0, 1$, *such that* $p_0 \ne p_1$ *and let* $\eta \in (0,1)$ *and* $\bar{\beta} \in \mathbb{R}^N$. *Put*

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1} \qquad \textit{and} \qquad \bar{\alpha} = (1-\eta)\bar{\alpha}^0 + \eta\bar{\alpha}^1 - \bar{\beta}.$$

Then

$$\big(L_{p_0,\bar{\alpha}^0,q_0}(\Omega), L_{p_1,\bar{\alpha}^1,q_1}(\Omega)\big)_{\eta,\bar{\beta},q} = L_{p,\bar{\alpha},q}(\Omega).$$

Proof. The proof is analogous to the proof of Theorem 2.4.2.

Let $r < \min(p_0, p_1)$ and choose $\theta_i \in (0,1)$ such that $\frac{1}{p_i} = \frac{1-\theta_i}{r}$, $i = 0, 1$. Put $\theta = (1-\eta)\theta_0 + \eta\theta_1$ and $\bar{\gamma} = (1-\eta)(-\bar{\alpha}^0) + \eta(-\bar{\alpha}^1) + \bar{\beta}$. Observe that $p = \frac{1-\theta}{r}$ and $\bar{\alpha} = -\bar{\gamma}$.

Now, by applying Proposition 2.5.2, the Reiteration Theorem (Corollary 2.5.1), and again Proposition 2.5.2, we get

$$\begin{aligned}\big(L_{p_0,\bar{\alpha}^0,q_0}(\Omega), L_{p_1,\bar{\alpha}^1,q_1}(\Omega)\big)_{\eta,\bar{\beta},q} &= \Big((L_r, L_\infty)_{\theta_0,-\bar{\alpha}^0,q_0}, (L_r, L_\infty)_{\theta_1,-\bar{\alpha}^1,q_1}\Big)_{\eta,\bar{\beta},q} \\ &= \big(L_r, L_\infty\big)_{\theta,\bar{\gamma},q} = L_{p,\bar{\alpha},q}.\end{aligned}$$

□

Remark 2.5.4. If we put $\bar{\alpha}^i = 0$ in Corollary 2.5.3, then we get

$$\big(L_{p_0,q_0}(\Omega), L_{p_1,q_1}(\Omega)\big)_{\eta,\bar{\beta},q} = L_{p,-\bar{\beta},q}(\Omega).$$

If we put $\bar{\beta} = 0$ as well, we receive the classical result stated in Theorem 2.1.5 above.

3 Characterization of Extrapolation Spaces

3.1 Δ-Extrapolation

Recall the logarithmic function parameters

$$\lambda_{\bar{\alpha}}(t) = l_1(t)^{\alpha_1} \cdots l_N(t)^{\alpha_N},$$

where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$, $l_1(t) = 1 + |\log t|$, and $l_i(t) = 1 + \log(l_{i-1}(t))$ for $i = 2, 3, \ldots$.

Here and in the following sections we will also use functions defined by

$$l_1^*(t) = t, \qquad l_2^*(t) = \log t, \qquad \ldots, \qquad l_i^*(t) = \log\big(l_{i-1}^*(t)\big),$$

and

$$\lambda_{\bar{\alpha}}^*(t) = \prod_{i=1}^{N} l_i^*(t)^{\alpha_i}, \qquad \bar{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N,$$

for all $t > 0$. There are numbers $t_i > 0$ such that, if we restrict the functions l_i^* to the sets (t_i, ∞), we have

$$l_1(2^t) \sim l_1^*(t), \qquad l_2(2^t) \sim l_2^*(t), \qquad \ldots, \qquad l_i(2^t) \sim l_i^*(t)$$

and

$$\lambda_{\bar{\alpha}}(2^t) \sim \lambda_{\bar{\alpha}}^*(t).$$

Definition 3.1.1. Let (A_0, A_1) be a compatible couple of quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < q \leq \infty$, and $\theta \in (0, 1)$.

(i) Let $\alpha > 0$ and $j_0 \in \mathbb{N}$ such that $\theta + 2^{-j_0} \in (0, 1)$. We define $\Delta^\alpha A_{\theta,q}$ to be the space of all $a \in \bigcap_{j=j_0}^{\infty} A_{\theta+2^{-j},q}$ with

$$\|a|\Delta^{\alpha} A_{\theta,q}\| := \left(\sum_{j=j_0}^{\infty} 2^{-j\alpha q} \|a|A_{\theta+2^{-j},q}\|^q \right)^{\frac{1}{q}} < \infty$$

if $q < \infty$. The space $\Delta^{\alpha} A_{\theta,\infty}$ is determined by

$$\|a|\Delta^{\alpha} A_{\theta,\infty}\| := \sup_{j \geq j_0} 2^{-j\alpha} \|a|A_{\theta+2^{-j},\infty}\| < \infty.$$

(ii) Let $N \in \mathbb{N}$, $(\alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{R}^{N-1}$, and $\alpha_N > 0$. We define $\Delta^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}$ to be the space of all $a \in \bigcap_{j=1}^{\infty} A_{\theta,\alpha_1,\ldots,\alpha_{N-1}+2^{-j},q}$ with

$$\|a|\Delta^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}\| := \left(\sum_{j=1}^{\infty} 2^{-j\alpha_N q} \|a|A_{\theta,\alpha_1,\ldots,\alpha_{N-1}+2^{-j},q}\|^q \right)^{\frac{1}{q}} < \infty.$$

The case $q = \infty$ is treated as in (i) above.

(iii) We put

$$\Delta^0 A_{\theta,q} := A_{\theta,q} \quad \text{and} \quad \Delta^0 A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q} := A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}.$$

Remark 3.1.1. In Theorem 2.5.1 we have seen that, if $A_0 \hookrightarrow A_1$, we have

$$A_{\theta,q} \hookrightarrow A_{\tilde{\theta},q}$$

for $0 < \theta < \tilde{\theta} < 1$ and

$$A_{\theta,\alpha_1,\ldots,\alpha_{N-2},\alpha_{N-1},q} \hookrightarrow A_{\theta,\alpha_1,\ldots,\alpha_{N-2},\tilde{\alpha}_{N-1},q}$$

for $\alpha_{N-1} < \tilde{\alpha}_{N-1}$. Therefore,

$$\bigcap_{j=j_0}^{\infty} A_{\theta+2^{-j},q} \quad \text{and} \quad \bigcap_{j=j_1}^{\infty} A_{\theta,\alpha_1,\ldots,\alpha_{N-1}+2^{-j},q} \tag{3.1.1}$$

do not depend on the starting points j_0 and j_1, respectively.

If we use couples (A_0, A_1) that are not ordered, the first intersection in (3.1.1) is different for different j_0. Consequently, the outcoming extrapolation spaces depend on j_0, as we will see in Section 3.3 below. In contrast to that, the monotonicity in α_{N-1} of the logarithmic spaces (with fixed θ and q) still holds, as we stated in Remark 2.5.3.

The following lemma contains two calculations which we will need in the proof of the next theorem.

Lemma 3.1.1. *Let* $0 < q \leq \infty$.

(i) *Let* $\varkappa > 0$. *Then*

$$\sum_{j=j_0}^{\infty} 2^{-j\varkappa q - t2^{-j}q} \sim t^{-\varkappa q}$$

for $t \in (1, \infty)$.

(ii) *Let* $N \in \mathbb{N}$ *and* $\varkappa > 0$. *Then*

$$\sum_{j=1}^{\infty} 2^{-j\varkappa q}\, l_N^*(t)^{-2^{-j}q} \sim l_{N+1}^*(t)^{-\varkappa q}$$

for $t \in (1, \infty)$.

Proof. The proof of (i) is taken from [5, p.237] although the same equivalence is proven in a different way in [18, pp. 81-82]. Choosing $t > 1$ and $j = k + [\log t]$ we get

$$\begin{aligned} \sum_{j=j_0}^{\infty} 2^{-j\varkappa q - t2^{-j}q} &= 2^{-[\log t]\varkappa q} \sum_{k=j_0-[\log t]}^{\infty} 2^{-k\varkappa q - \frac{t}{2^{[\log t]}}2^{-k}q} \\ &\sim t^{-\varkappa q} \sum_{k=j_0-[\log t]}^{\infty} 2^{-k\varkappa q - 2^{-k}q} \\ &\sim t^{-\varkappa q}. \end{aligned}$$

The last sum is equivalent to 1 because

$$\begin{aligned} 2^{-q} &\leq \sum_{k=1}^{[\log t]-j_0} 2^{k\varkappa q - 2^k q} + \sum_{k=0}^{\infty} 2^{-k\varkappa q - 2^{-k}q} \\ &\leq \sum_{k=1}^{\infty} 2^{k\varkappa q - 2^k q} + \sum_{k=0}^{\infty} 2^{-k\varkappa q} \leq c, \end{aligned}$$

where c is independent of t.

(ii) Choosing $t > 1$ and $j = k + [l^*_{N+2}(t)]$ we get

$$\begin{aligned}
\sum_{j=1}^{\infty} 2^{-j\varkappa q}\, l^*_N(t)^{-2^{-j}q} &\sim 2^{-l^*_{N+2}(t)\varkappa q} \sum_{k=1-[l^*_{N+2}(t)]}^{\infty} 2^{-k\varkappa q}\, l^*_N(t)^{-2^{-k}2^{-l^*_{N+2}(t)}q} \\
&= l^*_{N+1}(t)^{-\varkappa q} \sum_{k=1-[l^*_{N+2}(t)]}^{\infty} 2^{-k\varkappa q} \left(l^*_N(t)^{l^*_{N+1}(t)^{-1}} \right)^{-2^{-k}q} \\
&= l^*_{N+1}(t)^{-\varkappa q} \sum_{k=1-[l^*_{N+2}(t)]}^{\infty} 2^{-k\varkappa q}\, 2^{-2^{-k}q} \\
&\sim l^*_{N+1}(t)^{-\varkappa q}.
\end{aligned}$$

The last sum is the same one as in the proof of (i) and therefore equivalent to 1. □

Theorem 3.1.1. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < q \le \infty$ and $\theta \in (0,1)$.*

(i) *Let $\alpha \ge 0$. Then*

$$\Delta^{\alpha} A_{\theta,q} = A_{\theta,\alpha,q}.$$

(ii) *Let $N \in \mathbb{N}$, $N \ge 2$, $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{R}^{N-1}$, and $\alpha_N \ge 0$. Then*

$$\Delta^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q} = A_{\theta,\alpha_1,\ldots,\alpha_{N-1},\alpha_N,q}.$$

Proof. (i) The proof is taken from [5]. We assume $q < \infty$. The case $q = \infty$ follows by the same arguments.

For $\alpha = 0$ there is nothing to show. Let $\alpha > 0$. Using the definitions of the interpolation, logarithmic interpolation, and extrapolation methods and Lemma 3.1.1 (i) with $\varkappa = \alpha$ we get

$$\begin{aligned}
\|a|\Delta^{\alpha} A_{\theta,q}\|^q &= \sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q} \sum_{m=1}^{\infty} 2^{-m\theta q - m2^{-j}q} K(2^m, a)^q \\
&= \sum_{m=1}^{\infty} 2^{-m\theta q} K(2^m, a)^q \sum_{j=j_0}^{\infty} 2^{-j\alpha q} 2^{-m2^{-j}q} \\
&\sim \sum_{m=1}^{\infty} 2^{-m\theta q} m^{-\alpha q} K(2^m, a)^q \\
&= \|a|A_{\theta,\alpha,q}\|^q
\end{aligned}$$

(ii) Again, for $\alpha_N = 0$ there is nothing to prove. Let $\alpha_N > 0$. We use the equivalence of $l_i(2^t)$ and $l_i^*(t)$ and Lemma 3.1.1 (ii) with $\varkappa = \alpha_N$ to get

$$\begin{aligned}
&\sum_{j=1}^{\infty} 2^{-j\alpha_N q}\|a|A_{\theta,\alpha_1,\ldots,\alpha_{N-1}+2^{-j},q}\|^q \\
\sim\ &\sum_{j=1}^{\infty} 2^{-j\alpha_N q}\sum_{m=1}^{\infty} 2^{-m\theta q}\, l_1(2^m)^{-\alpha_1 q}\cdots l_{N-1}(2^m)^{-(\alpha_{N-1}+2^{-j})q}K(2^m,a)^q \\
=\ &\sum_{m=1}^{\infty} 2^{-m\theta q}\lambda_{\bar{\alpha}}(2^m)^{-q}K(2^m,a)^q\sum_{j=1}^{\infty} 2^{-j\alpha_N q}\, l_{N-1}(2^m)^{-2^{-j}q} \\
\sim\ &\sum_{m=1}^{\infty} 2^{-m\theta q}\lambda_{\bar{\alpha}}(2^m)^{-q}\, l_N(2^m)^{-\alpha_N q}\, K(2^m,a)^q \\
\sim\ &\|a|A_{\theta,\alpha_1,\ldots,\alpha_{N-1},\alpha_N,q}\|.
\end{aligned}$$

□

Definition 3.1.2. Let (A_0, A_1) be a compatible couple of quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < q \le \infty$, $\theta \in (0,1)$, and $j_0 \in \mathbb{N}$ such that $\theta + 2^{-j} \in (0,1)$ for all $j \ge j_0$. Let $N \in \mathbb{N}$, and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \ge 0$ for all $i = 1, \ldots, N$.

We define $\Delta^{\bar{\alpha}} A_{\theta,q}$ to be the space consisting of all $a \in \bigcap_{j=j_0}^{\infty} A_{\theta+2^{-j},q}$ with

$$\|a|\Delta^{\bar{\alpha}}A_{\theta,q}\| := \left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^{-q}\, \|a|A_{\theta+2^{-j},q}\|^q\right)^{\frac{1}{q}} < \infty.$$

Example 3.1.1. We choose $N = 2$ and $\bar{\alpha} = (\alpha_1, \alpha_2)$. Then

$$\begin{aligned}
\lambda_{\bar{\alpha}}\big(2^{2^j}\big) &= \big(1 + \log 2^{2^j}\big)^{\alpha_1}\big(1 + \log(1 + \log 2^{2^j})\big)^{\alpha_2} \\
&\sim 2^{j\alpha_1}(1 + \log 2^j)^{\alpha_2} \\
&\sim 2^{j\alpha_1} j^{\alpha_2}.
\end{aligned}$$

So, we have

$$\|a|\Delta^{(\alpha_1,\alpha_2)}A_{\theta,q}\| \sim \left(\sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, j^{-\alpha_2 q}\, \|a|A_{\theta+2^{-j},q}\|^q\right)^{\frac{1}{q}}.$$

Choosing $N = 1$ we recover Definition 3.1.1 (i).

Theorem 3.1.2. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces with*

$A_0 \hookrightarrow A_1$. *Let q, θ, N, and $\bar{\alpha}$ as in the above definition. Then*

$$\Delta^{\bar{\alpha}} A_{\theta,q} = A_{\theta,\bar{\alpha},q}.$$

Proof. We prove the theorem by induction. For $N = 1$ the assertion is true, as we proved in Theorem 3.1.1 (i). Assume that the assertion holds for $N \in \mathbb{N}$. If $\alpha_{N+1} > 0$, then, using Theorem 3.1.1 (ii), we can write

$$\begin{aligned}
&\|a|A_{\theta,\alpha_1,\ldots,\alpha_{N+1},q}\|^q \\
\sim\ &\|a|\Delta^{\alpha_{N+1}} A_{\theta,\alpha_1,\ldots,\alpha_N,q}\|^q \\
\sim\ &\sum_{k=1}^{\infty} 2^{-k\alpha_{N+1}q} \, \|a|A_{\theta,\alpha_1,\ldots,\alpha_{N-1},\alpha_N+2^{-k},q}\|^q \\
\sim\ &\sum_{k=1}^{\infty} 2^{-k\alpha_{N+1}q} \sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-(\alpha_N+2^{-k})q} \, \|a|A_{\theta+2^{-j},q}\|^q \\
\sim\ &\sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_1 q} \ldots l_{N-1}(2^j)^{-\alpha_N q} \left(\sum_{k=1}^{\infty} 2^{-k\alpha_{N+1}q}\, l_{N-1}(2^j)^{-2^{-k}q} \right) \|a|A_{\theta+2^{-j},q}\|^q \\
\sim\ &\sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_1 q} \ldots l_{N-1}(2^j)^{-\alpha_N q}\, l_N(2^j)^{-\alpha_{N+1}q} \, \|a|A_{\theta+2^{-j},q}\|^q.
\end{aligned}$$

For the sum in the big parenthesis we used Lemma 3.1.1 (put $l_0^*(j) = 2^j$ in case $N = 1$).

If $\alpha_{N+1} = 0$, then

$$\begin{aligned}
&\|a|A_{\theta,\alpha_1,\ldots,\alpha_{N+1},q}\|^q \sim \|a|A_{\theta,\alpha_1,\ldots,\alpha_N,q}\|^q \\
\sim\ &\sum_{j=1}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-\alpha_N q}\, l_N(2^j)^{-0\cdot q} \, \|a|A_{\theta+2^{-j},q}\|^q.
\end{aligned}$$

□

In Definition 3.1.1 we deal with interpolation spaces $A_{\theta+2^{-j},q}$. In applications we need q depending on j. Now we show that this is possible and leads to the same logarithmic interpolation spaces.

Theorem 3.1.3. *Let A_0 and A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < q \le \infty$, $0 < \theta < 1$, $r > 0$, and let $j_0 = j_0(\theta) \in \mathbb{N}$ such that, for all $j \ge j_0$, we have $\theta + 2^{-j} \in (0,1)$. Let $N \in \mathbb{N}$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \ge 0$ for all $i = 1, \ldots, N$.*

Put $\frac{1}{s_j} = \frac{1}{q} + \frac{1}{r2^j}$. *Then*

$$\|a|\Delta^{\bar{\alpha}} A_{\theta,q}\| \sim \left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}(2^{2^j})^{-q}\,\|a|A_{\theta+2^{-j},s_j}\|^q\right)^{\frac{1}{q}}.$$

Proof. Because $s_j < q$ we get for all $j \geq j_0$

$$\begin{aligned}\|a|A_{\theta+2^{-j},q}\| &= \left(\sum_{m=1}^{\infty} 2^{-mq(\theta+2^{-j})}K(2^m,a)^q\right)^{1/q}\\ &\leq \left(\sum_{m=1}^{\infty} 2^{-ms_j(\theta+2^{-j})}K(2^m,a)^{s_j}\right)^{1/s_j} = \|a|A_{\theta+2^{-j},s_j}\|.\end{aligned}$$

This implies

$$\|a|\Delta^{\bar{\alpha}} A_{\theta,q}\| \leq \left(\sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-\alpha_N q}\,\|a|A_{\theta+2^{-j},s_j}\|^q\right)^{\frac{1}{q}}.$$

Conversely, using Hölder's inequality, we get

$$\begin{aligned}\|a|A_{\theta+2^{-j},s_j}\| &= \left(\sum_{m=1}^{\infty} 2^{-ms_j(\theta+2^{-j})}K(2^m,a)^{s_j}\right)^{1/s_j}\\ &= \left(\sum_{m=1}^{\infty} 2^{-ms_j(\theta+2^{-j-1})}K(2^m,a)^{s_j}\cdot 2^{ms_j(2^{-j}-2^{-j-1})}\right)^{\frac{1}{s_j}}\\ &\leq \|a|A_{\theta+2^{-j-1},q}\|\left(\sum_{m=1}^{\infty} 2^{-mr/2}\right)^{\frac{1}{r2^j}}\\ &\leq c\|a|A_{\theta+2^{-j-1},q}\|.\end{aligned}$$

The function $l_1(2^j)^{-\alpha_2}\cdots l_{N-1}(2^j)^{-\alpha_N}$ is equivalent to a slowly varying function b. Theorem 1.4.1 (ii) implies $b(j-1) = b(\frac{j-1}{j}j) \leq c\frac{j}{j-1}b(j)$. With that and with the above estimate we get

$$
\begin{aligned}
&\sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-\alpha_N q}\, \|a|A_{\theta+2^{-j},s_j}\|^q \\
&\leq c \sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-\alpha_N q}\, \|a|A_{\theta+2^{-j-1},q}\|^q \\
&\leq c \sum_{j=j_0+1}^{\infty} 2^{-(j-1)\alpha_1 q} b(j-1)^q\, \|a|A_{\theta+2^{-j},q}\|^q \\
&\leq c 2^{\alpha_1 q} \sup_{j\geq j_0} \frac{j}{j-1} \sum_{j=j_0+1}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-\alpha_N q}\, \|a|A_{\theta+2^{-j},q}\|^q \\
&= C\|a|\Delta^{\bar{\alpha}} A_{\theta,q}\|^q.
\end{aligned}
$$

□

3.2 Σ-Extrapolation

Definition 3.2.1. Let (A_0, A_1) be a compatible couple of quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < \theta < 1$ and $0 < q \leq \infty$.

(i) Let $\alpha < 0$ and $j_0 \in \mathbb{N}$ such that $\theta - 2^{-j_0} \in (0,1)$. We let $\Sigma^\alpha A_{\theta,q}$ be the space of all $a \in A_1$ with $a = \sum_{j=j_0}^{\infty} a_j$ (convergence in A_1) and $a_j \in A_{\theta-2^{-j},q}$ such that

$$\left(\sum_{j=j_0}^{\infty} 2^{-j\alpha q}\|a_j|A_{\theta-2^{-j},q}\|^q\right)^{\frac{1}{q}} < \infty. \tag{3.2.1}$$

We put

$$\|a|\Sigma^\alpha A_{\theta,q}\| = \inf\left(\sum_{j=j_0}^{\infty} 2^{-j\alpha q}\|a_j|A_{\theta-2^{-j},q}\|^q\right)^{\frac{1}{q}},$$

where the infimum is taken over all decompositions with (3.2.1). In the case $q = \infty$, the definition is modified in the usual way.

(ii) Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha_1, \ldots, \alpha_{N-1} \in \mathbb{R}$, and $\alpha_N < 0$. We let $\Sigma^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}$ be the space of all $a \in A_1$ with $a = \sum_{j=1}^{\infty} a_j$ (convergence in A_1) and $a_j \in$

$A_{\theta,\alpha_1,\ldots,\alpha_{N-2},\alpha_{N-1}-2^{-j},q}$ such that

$$\left(\sum_{j=1}^{\infty} 2^{-j\alpha_N q} \|a_j | A_{\theta,\alpha_1,\ldots,\alpha_{N-2},\alpha_{N-1}-2^{-j},q}\|^q\right)^{\frac{1}{q}} < \infty. \tag{3.2.2}$$

We put

$$\|a|\Sigma^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}\| = \inf\left(\sum_{j=1}^{\infty} 2^{-j\alpha_N q} \|a_j | A_{\theta,\alpha_1,\ldots,\alpha_{N-2},\alpha_{N-1}-2^{-j},q}\|^q\right)^{\frac{1}{q}},$$

where the infimum is taken over all decompositions with (3.2.2). If $q = \infty$, use the usual modification.

(iii) We put

$$\Sigma^0 A_{\theta,q} := A_{\theta,q} \quad \text{and} \quad \Sigma^0 A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q} := A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}.$$

Theorem 3.2.1. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < q \leq \infty$, and $\theta \in (0,1)$.*

(i) *Let $\alpha \leq 0$. Then*
$$\Sigma^\alpha A_{\theta,q} = A_{\theta,\alpha,q}.$$

(ii) *Let $N \in \mathbb{N}$, $N \geq 2$, $\alpha_1, \ldots, \alpha_{N-1} \in \mathbb{R}$, and $\alpha_N \leq 0$. Then*

$$\Sigma^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q} = A_{\theta,\alpha_1,\ldots,\alpha_N,q}.$$

Proof. (i) The proof is taken from [4, pp. 69-71]. We use the J-Method of Real Interpolation. For given $a \in \Sigma^\alpha A_{\theta,q}$ and $\varepsilon > 0$ we can choose a representation $a = \sum_{j=j_0}^{\infty} a_j$ with $a_j \in A_{\theta-2^{-j},q}$ such that

$$\sum_{j=j_0}^{\infty} 2^{-j\alpha q} \|a_j | A_{\theta-2^{-j},q}\|^q \leq (1+\varepsilon)\|a|\Sigma^\alpha A_{\theta,q}\|^q, \tag{3.2.3}$$

using Definition 3.2.1. Now we decompose $a_j = \sum_{m=1}^{\infty} a_j^m$, $j \geq j_0$, such that $a_j^m \in A_0$ and

$$\sum_{m=1}^{\infty} 2^{-mq(\theta-2^{-j})} J(2^m, a_j^m)^q \leq (1+\varepsilon)\|a_j | A_{\theta-2^{-j},q}\|^q, \tag{3.2.4}$$

see the definition of the J-method.

Let c_j be the constant in the triangle inequality of A_j $(j = 0, 1)$. We put $c = \max(c_0, c_1)$ and let r be defined by $(2c)^r = 2$. We can assume that c is large, so that $r < q$. Let s be the number with $\frac{1}{q} + \frac{1}{s} = \frac{1}{r}$. Now we use Hölder's inequality to get

$$\begin{aligned}\left(\sum_{j=j_0}^{\infty} J(2^m, a_j^m)^r\right)^{\frac{1}{r}} &\leq \left(\sum_{j=j_0}^{\infty} 2^{-mq(\theta-2^{-j})-j\alpha q} J(2^m, a_j^m)^q\right)^{\frac{1}{q}} \left(\sum_{j=j_0}^{\infty} 2^{ms(\theta-2^{-j})+j\alpha s}\right)^{\frac{1}{s}} \\ &\sim 2^{m\theta} m^{\alpha} \left(\sum_{j=j_0}^{\infty} 2^{-mq(\theta-2^{-j})-j\alpha q} J(2^m, a_j^m)^q\right)^{\frac{1}{q}},\end{aligned} \tag{3.2.5}$$

where the last equivalence follows from Lemma 3.1.1 (i).

The last sum in (3.2.5) is finite due to the choice of the a_j^m, see (3.2.3) and (3.2.4). Since $J(2^m, \cdot)$ is a c-norm, it follows from (3.2.5) and Lemma 1.2.2 that $a^m := \sum_{j=j_0}^{\infty} a_j^m$ is convergent in A_0 and

$$\begin{aligned}J(2^m, a^m) &\leq \left(\sum_{j=j_0}^{\infty} J(2^m, a_j^m)^r\right)^{\frac{1}{r}} \\ &\leq C 2^{m\theta} m^{\alpha} \left(\sum_{j=j_0}^{\infty} 2^{-mq(\theta-2^{-j})-j\alpha q} J(2^m, a_j^m)^q\right)^{\frac{1}{q}}.\end{aligned}$$

Therefore we have $a = \sum_{m=1}^{\infty} a^m$ with

$$\begin{aligned}\|a|A_{\theta,\alpha,q}\|^q &\leq \sum_{m=1}^{\infty} 2^{-m\theta q}(1 + \log 2^m)^{-\alpha q} J(2^m, a^m)^q \\ &\sim \sum_{m=1}^{\infty} 2^{-m\theta q} m^{-\alpha q} J(2^m, a^m)^q \\ &\leq C^q \sum_{j=j_0}^{\infty} 2^{-j\alpha q} \sum_{m=1}^{\infty} 2^{-mq(\theta-2^{-j})} J(2^m, a_j^m)^q \\ &\leq C^q(1+\varepsilon) \sum_{j=j_0}^{\infty} 2^{-j\alpha q} \|a_j|A_{\theta-2^{-j},q}\|^q \\ &\leq C^q(1+\varepsilon)^2 \|a|\Sigma^{\alpha} A_{\theta,q}\|^q.\end{aligned}$$

To prove the converse embedding we take $a \in A_{\theta,\alpha,q}$ and choose a representation

$a = \sum_{m=1}^{\infty} a_m$ with $a_m \in A_0$ such that $\sum_{m=1}^{\infty} 2^{-m\theta q}(1 + \log 2^m)^{-\alpha q} J(2^m, a_m)^q < \infty$. If we put

$$a^j = \sum_{m=2^{j-j_0}}^{2^{j-j_0+1}-1} a_m$$

for $j \geq j_0$, then we get $a^j \in A_{\theta - 2^{-j},q}$ and $a = \sum_{j=j_0}^{\infty} a^j$.

Now

$$\begin{aligned}
\|a|\Sigma^\alpha A_{\theta,q}\|^q &\leq \sum_{j=j_0}^{\infty} 2^{-j\alpha q} \|a^j|A_{\theta-2^{-j},q}\|^q \\
&\leq \sum_{j=j_0}^{\infty} 2^{-j\alpha q} \sum_{m=2^{j-j_0}}^{2^{j-j_0+1}-1} 2^{-mq(\theta-2^{-j})} J(2^m, a_m)^q \\
&\sim \sum_{m=1}^{\infty} 2^{-mq\theta} m^{-\alpha q} J(2^m, a_m)^q.
\end{aligned}$$

Now, taking the infimum over all admitted representations we see that $A_{\theta,\alpha,q} \hookrightarrow \Sigma^\alpha A_{\theta,q}$.

(ii) The argumentation is analogous to (i). Let $a \in \Sigma^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}$. For $\varepsilon > 0$ we can choose a representation $a = \sum_{j=j_0}^{\infty} a_j$ with $a_j \in A_{\theta,\alpha_1,\ldots,\alpha_{N-2},\alpha_{N-1}-2^{-j},q}$ such that

$$\sum_{j=1}^{\infty} 2^{-j\alpha_N q} \|a_j|A_{\theta,\alpha_1,\ldots,\alpha_{N-2},\alpha_{N-1}-2^{-j},q}\|^q \leq (1+\varepsilon)\|a|\Sigma^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}\|^q.$$

Now we decompose $a_j = \sum_{m=1}^{\infty} a_j^m$, $j \geq j_0$, such that $a_j^m \in A_0$ and

$$\sum_{m=1}^{\infty} 2^{-m\theta q}\, l_1(2^m)^{-\alpha_1 q} \cdots l_{N-2}(2^m)^{-\alpha_{N-2} q}\, l_{N-1}(2^m)^{-(\alpha_{N-1}-2^{-j})q} J(2^m, a_j^m)^q \\
\leq (1+\varepsilon)\|a_j|A_{\theta,\alpha_1,\ldots,\alpha_{N-2},\alpha_{N-1}-2^{-j},q}\|^q.$$

We define r and s as in the proof of (i). We proceed as in (i), using Lemma 3.1.1 (ii) to get that $a^m := \sum\limits_{j=j_0}^{\infty} a_j^m$ is convergent and $a = \sum\limits_{m=1}^{\infty} a^m$. Then

$$
\begin{aligned}
&\|a|A_{\theta,\alpha_1,\ldots,\alpha_N,q}\|^q \\
&\leq \sum_{m=1}^{\infty} 2^{-m\theta q} l_1(2^m)^{-\alpha_1 q} \cdots l_N(2^m)^{-\alpha_N q} J(2^m, a^m)^q \\
&\leq C \sum_{j=j_0}^{\infty} 2^{-j\alpha_N q} \sum_{m=1}^{\infty} 2^{-m\theta q} l_1(2^m)^{-\alpha_1 q} \cdots l_{N-1}(2^m)^{-(\alpha_{N-1}-2^{-j})q}\, J(2^m, a_j^m)^q \\
&\leq C(1+\varepsilon) \sum_{j=j_0}^{\infty} 2^{-j\alpha_N q}\, \|a_j|A_{\theta,\alpha_1,\ldots,\alpha_{N-1}-2^{-j},q}\|^q \\
&\leq C(1+\varepsilon)^2\, \|a|\Sigma^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}\|^q.
\end{aligned}
$$

Conversely, let $a \in A_{\theta,\alpha_1,\ldots,\alpha_N,q}$ and choose a representation $a = \sum\limits_{m=1}^{\infty} a_m$ with $a_m \in A_0$ such that $\sum\limits_{m=1}^{\infty} 2^{-m\theta q}\, l_1(2^m)^{-\alpha_1 q} \ldots, l_N(2^m)^{-\alpha_N q} J(2^m, a_m)^q < \infty$. To adapt the proof of (i) to the case of (ii), we define $\pi_N(t)$ for every $N \in \mathbb{N}$ and $t \geq 0$ by

$$
\pi_1(t) = t, \quad \pi_2(t) = 2^t, \quad \ldots, \quad \pi_N(t) = 2^{\pi_{N-1}(t)}.
$$

Note, that $\pi_N\big(l_N(2^t)\big) \sim t$ for all $N \in \mathbb{N}$. Now, we put

$$
a^j = \sum_{m=\pi_{N+1}(j-j_0)-\pi_{N+1}(0)+1}^{\pi_{N+1}(j-j_0+1)-\pi_{N+1}(0)} a_m
$$

for $j \geq j_0$. Then we get $a^j \in A_{\theta,\alpha_1,\ldots,\alpha_{N-1}-2^{-j},q}$ and $a = \sum\limits_{j=j_0}^{\infty} a^j$. Then,

$$
\begin{aligned}
&\|a|\Sigma^{\alpha_N} A_{\theta,\alpha_1,\ldots,\alpha_{N-1},q}\|^q \\
&\leq \sum_{j=j_0}^{\infty} 2^{-j\alpha_N q} \|a^j|A_{\theta,\alpha_1,\ldots,\alpha_{N-1}-2^{-j},q}\|^q \\
&\leq \sum_{j=j_0}^{\infty} 2^{-j\alpha_N q} \sum_{m \text{ as above}} 2^{-m\theta q}\, l_1(2^m)^{-\alpha_1 q} \cdots l_{N-1}(2^m)^{-\alpha_{N-1}-2^{-j}q}\, J(2^m, a_m)^q \\
&\sim \sum_{m=1}^{\infty} 2^{-mq\theta}\, l_1(2^m)^{-\alpha_1 q} \cdots l_N(2^m)^{-\alpha_N q}\, J(2^m, a_m)^q.
\end{aligned}
$$

To change the order of the summation, we have to replace 2^j by $l_N(2^m)$. Then $l_{N-1}(2^m)^{-2^{-j}q}\,2^{-j\alpha_N q} \sim c\,l_N(2^m)^{-\alpha_N q}$, which is what we used in the above calculation. Now, by taking the infimum over all admitted representations, the result follows. $\square$

Definition 3.2.2. Let (A_0, A_1) be a compatible couple of quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < q \le \infty$, $\theta \in (0,1)$, and $j_0 \in \mathbb{N}$ such that $\theta - 2^{-j} \in (0,1)$ for all $j \ge j_0$. Let $N \in \mathbb{N}$, and $\bar{\alpha} = (\alpha_1, \dots, \alpha_N)$ with $\alpha_i \le 0$ for all $i = 1, \dots, N$.

We define $\Sigma^{\bar{\alpha}} A_{\theta,q}$ to be the space consisting of all $a \in A_1$ with $\sum\limits_{j=j_0}^{\infty} a_j$ and $a_j \in A_{\theta-2^{-j},q}$ such that

$$\left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^{-q} \, \|a_j|A_{\theta-2^{-j},q}\|^q \right)^{\frac{1}{q}} < \infty.$$

We put

$$\|a|\Sigma^{\bar{\alpha}} A_{\theta,q}\| := \inf \left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^{-q} \, \|a_j|A_{\theta-2^{-j},q}\|^q \right)^{\frac{1}{q}}.$$

Theorem 3.2.2. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let q, θ, N, and $\bar{\alpha}$ as in the above definition. Then*

$$\Sigma^{\bar{\alpha}} A_{\theta,q} = A_{\theta,\bar{\alpha},q}.$$

Proof. We prove the theorem by induction. For $N = 1$ the assertion is true, as we proved in Theorem 3.2.1 (i). Assume that the assertion holds for $N \in \mathbb{N}$ and let $\alpha_{N+1} > 0$. The proof is similar to the proof of Theorem 3.2.1. Let $a \in A_{\theta,\alpha_1,\dots,\alpha_{N+1},q} = \Sigma^{\alpha_{N+1}} A_{\theta,\alpha_1,\dots,\alpha_N,q}$ and $\varepsilon > 0$. We choose a representation $a = \sum\limits_{k=1}^{\infty} a_k$ such that

$$\sum_{k=1}^{\infty} 2^{-k\alpha_{N+1}q} \, \|a_k|A_{\theta,\alpha_1,\dots,\alpha_N-2^{-k},q}\|^q \le (1+\varepsilon)\, \|a|\Sigma^{\alpha_{N+1}} A_{\theta,\alpha_1,\dots,\alpha_N,q}\|^q.$$

Now, for every k, we choose a representation $a_k = \sum\limits_{j=j_0}^{\infty} a_k^j$ such that

$$\begin{aligned}
&\sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-(\alpha_N - 2^{-k})q}\, \|a_k^j | A_{\theta-2^{-j},q}\|^q \\
&\le (1+\varepsilon)\, \|a_k | A_{\theta,\alpha_1,\ldots,\alpha_N - 2^{-k},q}\|^q \\
&\le c(1+\varepsilon)\, \|a_k | \Sigma^{(\alpha_1,\ldots,\alpha_N - 2^{-k})} A_{\theta,q}\|^q,
\end{aligned}$$

where we used the induction hypothesis. Now, as in the proof of Theorem 3.2.1, let c denote the constant in the quasi-triangle inequality such that r, defined by $(2c)^r = 2$, is smaller than q. Define $s > 0$ by $\frac{1}{q} + \frac{1}{s} = \frac{1}{r}$. Then, by Hölder's Inequality,

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty} \|a_k^j | A_{\theta-2^{-j},q}\|^r\right)^{\frac{1}{r}} \\
&\le \left(\sum_{k=1}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-(\alpha_N - 2^{-k})q}\, 2^{-k\alpha_{N+1} q}\, \|a_k^j | A_{\theta-2^{-j},q}\|^q\right)^{\frac{1}{q}} \cdot \\
&\qquad \left(\sum_{k=1}^{\infty} 2^{j\alpha_1 s}\, l_1(2^j)^{\alpha_2 s} \cdots l_{N-1}(2^j)^{(\alpha_N - 2^{-k})s}\, 2^{k\alpha_{N+1} s}\right)^{\frac{1}{s}} \\
&\le c\, 2^{j\alpha_1}\, l_1(2^j)^{\alpha_2} \cdots l_N(2^j)^{\alpha_{N+1}} \cdot \\
&\qquad \left(\sum_{k=1}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-(\alpha_N - 2^{-k})q}\, 2^{-k\alpha_{N+1} q}\, \|a_k^j | A_{\theta-2^{-j},q}\|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where we applied Lemma 3.1.1 (ii). Now put $a^j := \sum\limits_{k=1}^{\infty} a_k^j$. Then

$$\begin{aligned}
&\|a^j | A_{\theta-2^{-j},q}\| \\
&\le \left(\sum_{k=1}^{\infty} \|a_k^j | A_{\theta-2^{-j},q}\|^r\right)^{\frac{1}{r}} \\
&\le c 2^{j\alpha_1}\, l_1(2^j)^{\alpha_2} \cdots l_N(2^j)^{\alpha_{N+1}} \cdot \\
&\qquad \left(\sum_{k=1}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-(\alpha_N - 2^{-k})q}\, 2^{-k\alpha_{N+1} q}\, \|a_k^j | A_{\theta-2^{-j},q}\|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

It follows that $a = \sum\limits_{j=j_0}^{\infty} a^j$ and

$$
\begin{aligned}
&\|a|\Sigma^{(\alpha_1,\dots,\alpha_{N+1})}A_{\theta,q}\|^q \\
&\leq \sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_N(2^j)^{-\alpha_{N+1} q}\, \|a^j|A_{\theta-2^{-j},q}\|^q \\
&\leq c \sum_{j=j_0}^{\infty} \sum_{k=1}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-(\alpha_N - 2^{-k})q}\, 2^{-k\alpha_{N+1} q}\, \|a_k^j|A_{\theta-2^{-j},q}\|^q \\
&\leq c(1+\varepsilon) \sum_{k=1}^{\infty} 2^{-k\alpha_{N+1} q}\, \|a_k|\Sigma^{(\alpha_1,\dots,\alpha_N - 2^{-k})}A_{\theta,q}\|^q \\
&\leq c(1+\varepsilon)^2\, \|a|\Sigma^{\alpha_{N+1}}A_{\theta,\alpha_1,\dots,\alpha_N,q}\|^q \\
&\leq c\, \|a|A_{\theta,\alpha_1,\dots,\alpha_{N+1},q}\|^q.
\end{aligned}
$$

Now, we prove the converse inequality. Let $a \in \Sigma^{(\alpha_1,\dots,\alpha_{N+1})}A_{\theta,q}$. Let $a = \sum\limits_{j=j_0}^{\infty} a_j$ be a decomposition such that

$$
\sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_N(2^j)^{-\alpha_{N+1} q}\, \|a_j|A_{\theta-2^{-j},q}\|^q < \infty.
$$

We use π_N from the proof of Theorem 3.2.1 (ii) and put

$$
a^k = \sum_{j=\pi_{N+1}(k-1)-\pi_{N+1}(0)+j_0}^{\pi_{N+1}(k)-\pi_{N+1}(0)+j_0-1} a_j.
$$

Then,

$$
\begin{aligned}
&\|a|A_{\theta,\alpha_1,\dots,\alpha_{N+1},q}\|^q \\
&\leq c\|a|\Sigma^{\alpha_{N+1}}A_{\theta,\alpha_1,\dots,\alpha_N,q}\|^q \\
&\leq c \sum_{k=1}^{\infty} 2^{-k\alpha_{N+1} q}\|a^k|\Sigma^{(\alpha_1,\dots,\alpha_N - 2^{-k})}A_{\theta,q}\|^q \\
&\leq c \sum_{k=1}^{\infty} 2^{-k\alpha_{N+1} q} \sum_{j \text{ as above}} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-(\alpha_N - 2^{-k})q}\, \|a_j|A_{\theta-2^{-j},q}\|^q \\
&= c \sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_N(2^j)^{-\alpha_{N+1} q}\, \|a_j|A_{\theta-2^{-j},q}\|^q,
\end{aligned}
$$

where we used $2^{-k\alpha_{N+1}q}\, l_{N-1}(2^j)^{2^{-k}q} \sim c\, l_N(2^j)^{-\alpha_{N+1}q}$, when changing the summation order by putting $2^k = l_N(2^j)$. Now, take the infimum and the proof is complete. □

The next theorem is the Σ-counterpart of Theorem 3.1.3. It is a generalization of [4, Th. 2.4].

Theorem 3.2.3. *Let A_0 and A_1 be a compatible couple of quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < q < \infty$, $0 < \theta < 1$, $r > 0$, and let $j_0 = j_0(\theta) \in \mathbb{N}$ such that, for all $j \geq j_0$, we have $\theta - 2^{-j} \in (0,1)$. Let $N \in \mathbb{N}$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \leq 0$ for all $i = 1, \ldots, N$. Put $\frac{1}{u_j} = \frac{1}{q} - \frac{1}{r2^j}$. Then the space $\Sigma^{\bar{\alpha}} A_{\theta,q}$ consists of all $a \in A_1$ that have a representation $a = \sum\limits_{j=j_0}^{\infty} a_j$ (convergence in A_1) such that*

$$\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^{-q} \, \|a_j | A_{\theta-2^{-j},u_j}\|^q < \infty.$$

The infimum of this expression, taken over all admissible representations, is an equivalent norm in $\Sigma^{\bar{\alpha}} A_{\theta,q}$.

Proof. Let $a \in \Sigma^{\bar{\alpha}} A_{\theta,q}$ with $a = \sum\limits_{j=j_0}^{\infty} a_j$, $a_j \in A_{\theta-2^{-j},q}$, such that

$$\sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-\alpha_N q} \, \|a_j | A_{\theta-2^{-j},q}\|^q < \infty.$$

Because $q < u_j$, we have $\|a_j | A_{\theta-2^{-j},u_j}\| \leq \|a_j | A_{\theta-2^{-j},q}\|$. So, $a_j \in A_{\theta-2^{-j},u_j}$ and

$$\sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\, l_1(2^j)^{-\alpha_2 q} \cdots l_{N-1}(2^j)^{-\alpha_N q} \, \|a_j | A_{\theta-2^{-j},u_j}\|^q < \infty.$$

Now we prove the converse inequality. For $j \geq j_0$ and $a_j \in A_{\theta-2^{-j},u_j}$ holds

$$\begin{aligned}
\|a_j|A_{\theta-2^{-j-1},q}\| &= \left(\sum_{m=1}^{\infty} 2^{-m(\theta-2^{-j-1})q}\,K(2^m,a_j)^q\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{m=1}^{\infty} 2^{-m(\theta-2^{-j})u_j}\,K(2^m,a_j)^{u_j}\right)^{\frac{1}{u_j}} \left(\sum_{m=1}^{\infty} 2^{m(2^{-j-1}-2^{-j})r2^j}\right)^{\frac{1}{r2^j}} \\
&= \|a_j|A_{\theta-2^{-j},u_j}\| \left(\sum_{m=1}^{\infty} 2^{-\frac{1}{2}mr}\right)^{\frac{1}{r2^j}} \\
&\leq c\,\|a_j|A_{\theta-2^{-j},u_j}\|.
\end{aligned}$$

As in Theorem 3.1.3, this implies

$$\|a|\Sigma^{\bar{\alpha}}A_{\theta,q}\| \leq C\,\inf \sum_{j=j_0}^{\infty} 2^{-j\alpha_1 q}\,l_1(2^j)^{-\alpha_2 q}\cdots l_{N-1}(2^j)^{-\alpha_N q}\,\|a_j|A_{\theta-2^{-j},u_j}\|^q.$$

□

3.3 Variations

We now want to examine some variations of the methods used above. To be precise, we will take a pair (A_0, A_1) that is not necessarily ordered and do the approach to the parameter from the other side. We do not get the same results, but we can characterize the outcoming extrapolation spaces. We mix ideas from [4] and [18]. The latter paper deals with a more general setting. To avoid confusion please note that we do not use the same notation as in [18], see Remark 3.3.1 below.

In this section, we will only give a description of the Δ method. Analogous results can be proved for the Σ method, see [18].

Definition 3.3.1. Let (A_0, A_1) be a compatible pair of quasi-Banach spaces. Let $0 < q \leq \infty$, $\theta \in (0,1)$, and $\alpha > 0$.

(i) Let $j_0 \in \mathbb{N}$ such that $\theta + 2^{-j_0} \in (0,1)$. We define $\Delta^{\alpha}_{j_0} A_{\theta+,q}$ to be the space of all $a \in \bigcap_{j=j_0}^{\infty} A_{\theta+2^{-j},q}$ with

$$\|a|\Delta^{\alpha}_{j_0} A_{\theta+,q}\| = \left(\sum_{j=j_0}^{\infty} 2^{-j\alpha q}\|a|A_{\theta+2^{-j},q}\|^q\right)^{\frac{1}{q}} < \infty.$$

(ii) Analogously, for $j_0 \in \mathbb{N}$ with $\theta - 2^{-j_0} \in (0,1)$, we let $\Delta^\alpha_{j_0} A_{\theta-,q}$ be the space of all $a \in \bigcap_{j=j_0}^{\infty} A_{\theta-2^{-j},q}$ with

$$\|a|\Delta^\alpha_{j_0} A_{\theta-,q}\| = \left(\sum_{j=j_0}^{\infty} 2^{-j\alpha q} \|a|A_{\theta-2^{-j},q}\|^q \right)^{\frac{1}{q}} < \infty.$$

Remark 3.3.1. Our Δ-methods are special cases of the $\delta^{(q)}$-methods in [18], more precisely,

$$\Delta^\alpha_{j_0} A_{\theta+,q} = \delta^{(q)-}_{\theta,\beta}(M(\eta)A_{\eta,q}),$$

where $\eta = \theta + 2^{-j}$, $\beta = \theta + 2^{-j_0}$, and $M(\eta) = (\eta - \theta)^\alpha$ and

$$\Delta^\alpha_{j_0} A_{\theta-,q} = \delta^{(q)+}_{\beta,\theta}(M(\eta)A_{\eta,q}),$$

where $\eta = \theta - 2^{-j}$, $\beta = \theta - 2^{-j_0}$, and $M(\eta) = (\theta - \eta)^\alpha$.

The following lemma contains an elementary calculation which we will need in the proof of the next theorem.

Lemma 3.3.1. *Let $\alpha > 0$ and $0 < q \leq \infty$. Then*

$$\sum_{j=j_0}^{\infty} 2^{-j\alpha q + t2^{-j}q} \sim 2^{tq2^{-j_0}} \tag{3.3.1}$$

for $t \in (1, \infty)$.

Proof. The proof is taken from [18, pp.82-83]. It holds

$$\begin{aligned}
2^{-j_0\alpha q} 2^{tq2^{-j_0}} &\leq \sum_{j=j_0}^{\infty} 2^{-j\alpha q + t2^{-j}q} \\
&= 2^{-j_0\alpha q} \sum_{k=0}^{\infty} 2^{-k\alpha q + tq2^{-k}2^{-j_0}} \\
&= 2^{-j_0\alpha q + tq2^{-j_0}} \sum_{k=0}^{\infty} 2^{-k\alpha q + tq2^{-j_0}(2^{-k}-1)} \\
&\leq 2^{-j_0\alpha q} 2^{tq2^{-j_0}} \sum_{k=0}^{\infty} 2^{-k\alpha q} \\
&\leq c 2^{tq2^{-j_0}},
\end{aligned}$$

where c does not depend on t. □

Theorem 3.3.1. *Let (A_0, A_1) be a compatible couple of quasi-Banach spaces. Let $0 < q \leq \infty$, $\theta \in (0,1)$, and $\alpha > 0$.*

(i) *Let $j_0 \in \mathbb{N}$ such that $\theta + 2^{-j_0} \in (0,1)$. Then*

$$\Delta^{\alpha}_{j_0} A_{\theta+,q} = A_{\theta+2^{-j_0},q} \cap A_{\theta,\alpha,q}.$$

(ii) *For $j_0 \in \mathbb{N}$ with $\theta - 2^{-j_0} \in (0,1)$, we have*

$$\Delta^{\alpha}_{j_0} A_{\theta-,q} = A_{\theta-2^{-j_0},q} \cap A_{\theta,\alpha,q}.$$

Proof. With the help of Lemma 3.3.1, the proof follows exactly the proof of Theorem 3.1.1 (i). □

If we now additionally assume that $A_0 \hookrightarrow A_1$, then we get the following result (see Theorem 3.1.1 and Remark 3.1.1).

Corollary 3.3.1. *Let (A_0, A_1) with $A_0 \hookrightarrow A_1$. Then,*

$$\Delta^{\alpha}_{j_0} A_{\theta+,q} = A_{\theta,\alpha,q}$$

and

$$\Delta^{\alpha}_{j_0} A_{\theta-,q} = A_{\theta-2^{-j_0},q}.$$

3.4 Applications to concrete function spaces

In this section we want to apply the above extrapolation results to generalized Lorentz-Zygmund spaces. We extend the assertions from [4, pp. 74-79].

At first, we characterize spaces where $q = p$. For that we apply Theorems 3.1.3 and 3.2.3 to get a characterization by Lebesgue spaces. Secondly, we treat the general case, where the characterization is made with Lorentz spaces.

Corollary 3.4.1. *Let $\Omega \subseteq \mathbb{R}^n$ with finite Lebesgue measure. Let $0 < p < \infty$ and let $j_0 = j_0(p) \in \mathbb{N}$ such that, for all $j \geq j_0$, we have $\frac{1}{p^{\lambda_j}} := \frac{1}{p} - \frac{1}{n2^j} > 0$. Put $\frac{1}{p^{\sigma_j}} := \frac{1}{p} + \frac{1}{n2^j}$.*

(i) *Let* $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$ *with* $\alpha_i \le 0$, $i = 1, \ldots, N$. *Then*

$$\|f|L_{p,\bar{\alpha}}(\Omega)\| \sim \left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^p \, \|f|L_{p^{\sigma_j}}(\Omega)\|^p\right)^{1/p}.$$

(ii) *Let* $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$ *with* $\alpha_i \ge 0$, $i = 1, \ldots, N$. *Then* $L_{p,\bar{\alpha}}(\Omega)$ *consists of all measurable functions* f *on* Ω *which can be represented as*

$$f = \sum_{j=j_0}^{\infty} f_j$$

where $f_j \in L_{p^{\lambda_j}}(\Omega)$ *such that*

$$\left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^p \, \|f|L_{p^{\lambda_j}}(\Omega)\|^p\right)^{1/p} < \infty.$$

The infimum of the last expression taken over all admissible representations is an equivalent quasi-norm in $L_{p,\bar{\alpha}}(\Omega)$.

Proof. Let $0 < r < p$ and let $\theta = \frac{r}{p}$. Because $|\Omega| < \infty$, we have $L_\infty(\Omega) \hookrightarrow L_r(\Omega)$. Applying Corollary 2.5.3 gives

$$(L_\infty(\Omega), L_r(\Omega))_{\theta,-\bar{\alpha},p} = L_{p,\bar{\alpha}}(\Omega).$$

With $\frac{1}{p_*^{\sigma_j}} := \frac{1}{p} + \frac{1}{r2^j}$ and $\frac{1}{p_*^{\lambda_j}} := \frac{1}{p} - \frac{1}{r2^j}$ it holds

$$(L_\infty(\Omega), L_r(\Omega))_{\theta+2^{-j},p_*^{\sigma_j}} = L_{p_*^{\sigma_j}}(\Omega)$$

and

$$(L_\infty(\Omega), L_r(\Omega))_{\theta-2^{-j},p_*^{\lambda_j}} = L_{p_*^{\lambda_j}}(\Omega).$$

If $\alpha_i \le 0$ for all i, we apply Theorems 3.1.2 and 3.1.3 and get

$$\|f|L_{p,\bar{\alpha}}(\Omega)\| \sim \left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^p \, \|f|L_{p_*^{\sigma_j}}(\Omega)\|^p\right)^{1/p}.$$

We may assume that r is smaller that n, which implies $p_*^{\sigma_j} < p^{\sigma_j}$ and, consequently, $\|f|L_{p_*^{\sigma_j}}(\Omega)\| \le \|f|L_{p^{\sigma_j}}(\Omega)\|$. Then we choose $j_1 \ge j_0$ sufficiently large such that $r2^{j+j_1} > n2^j$. That means $p_*^{\sigma_{j+j_1}} > p^{\sigma_j}$ and we get $\|f|L_{p^{\sigma_j}}(\Omega)\| \le \|f|L_{p_*^{\sigma_{j+j_1}}}(\Omega)\|$.

Now, (i) follows.

If $\alpha_i \geq 0$, it follows from Theorems 3.2.2 and 3.2.3 that $L_{p,\bar{\alpha}}(\Omega)$ consists of all f with a representation $f = \sum_{j=j_0}^{\infty} f_j$ where $f_j \in L_{p_*^{\lambda_j}}(\Omega)$ such that

$$\left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^p \, \|f|L_{p_*^{\lambda_j}}(\Omega)\|^p \right)^{1/p} < \infty.$$

As in the situation of (i), we can show that, if we replace $p_*^{\lambda_j}$ by p^{λ_j}, the quasi-norms are equivalent. □

Example 3.4.1. (i) We consider $N = 1$ and let $\bar{\alpha} = (\alpha)$. Then we have $\lambda_{\bar{\alpha}}\big(2^{2^j}\big) \sim 2^{j\alpha}$ and, if $\alpha < 0$,

$$\|f|L_p(\log L)_\alpha(\Omega)\| \sim \left(\sum_{j=j_0}^{\infty} 2^{j\alpha p} \, \|f|L_{p^{\sigma_j}}(\Omega)\|^p \right)^{1/p}.$$

If $\alpha < 0$ we get an analogous result from the second part of Corollary 3.4.1. This is the result from [4, pp. 74,75].

(ii) Next, we consider $N = 2$ and let $\bar{\alpha} = (0, \alpha)$ with $\alpha < 0$. Then $\lambda_{\bar{\alpha}}\big(2^{2^j}\big) \sim j^\alpha$. It follows

$$\|f|L_{p,0,\alpha}(\Omega)\| \sim \left(\sum_{j=j_0}^{\infty} j^{\alpha p} \, \|f|L_{p^{\sigma_j}}(\Omega)\|^p \right)^{1/p}.$$

An analogous result holds for $\alpha > 0$.

Corollary 3.4.2. *Let $\Omega \subseteq \mathbb{R}^n$ with finite Lebesgue measure. Let $0 < p < \infty$, $0 < q \leq \infty$, and let $j_0 = j_0(p) \in \mathbb{N}$ such that, for all $j \geq j_0$, we have $\frac{1}{p^{\nu_j}} := \frac{1}{p} - 2^{-j} > 0$. Put $\frac{1}{p^{\mu_j}} := \frac{1}{p} + 2^{-j}$.*

(i) *Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \leq 0$, $i = 1, \ldots, N$. Then*

$$\|f|L_{p,\bar{\alpha},q}(\Omega)\| \sim \left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^q \, \|f|L_{p^{\mu_j},q}(\Omega)\|^q \right)^{1/q}.$$

(ii) *Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \geq 0$, $i = 1, \ldots, N$. Then $L_{p,\bar{\alpha},q}(\Omega)$ consists of all measurable functions f on Ω that can be represented as $f = \sum_{j=j_0}^{\infty} f_j$ where*

$f_j \in L_{p^{\nu_j},q}(\Omega)$ such that

$$\left(\sum_{j=j_0}^{\infty} \lambda_{\bar{\alpha}}\big(2^{2^j}\big)^q \, \|f|L_{p^{\nu_j},q}(\Omega)\|^q\right)^{1/q} < \infty.$$

The infimum of the last expression taken over all admissible representations is a equivalent quasi-norm in $L_{p,\bar{\alpha},q}(\Omega)$.

Proof. Take $0 < r < \min(1, p, q)$ and let $\theta = \frac{r}{p}$. Applying Corollary 2.5.3, we get

$$\big(L_\infty(\Omega), L_r(\Omega)\big)_{\theta,-\bar{\alpha},q} = L_{p,\bar{\alpha},q}(\Omega).$$

Put $\dfrac{1}{p^{\eta_j}} = \frac{1}{p} + \frac{1}{r2^j}$ and $\dfrac{1}{p^{\tau_j}} = \frac{1}{p} - \frac{1}{r2^j}$. Then

$$\big(L_\infty(\Omega), L_r(\Omega)\big)_{\theta+2^{-j},q} = L_{p^{\eta_j},q}(\Omega)$$

and

$$\big(L_\infty(\Omega), L_r(\Omega)\big)_{\theta-2^{-j},q} = L_{p^{\tau_j},q}(\Omega).$$

Because $\dfrac{1}{p^{\eta_j}} - \dfrac{1}{p^{\mu_j}} = \dfrac{1-r}{r2^j}$ tends to zero as $j \to \infty$, it holds

$$\begin{aligned}
\|f|L_{p^{\eta_j},q}(\Omega)\| &= \left(\int_0^{|\Omega|} \big[t^{\frac{1}{p^{\eta_j}}} f^*(t)\big]^q \, \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\le \sup_{0<t<|\Omega|} \big\{t^{\frac{1-r}{r2^j}}\big\} \left(\int_0^{|\Omega|} \big[t^{\frac{1}{p^{\mu_j}}} f^*(t)\big]^q \, \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\le |\Omega|^{\frac{1-r}{r2^j}} \, \|f|L_{p^{\mu_j},q}(\Omega)\| \\
&\le M\|f|L_{p^{\mu_j},q}(\Omega)\|.
\end{aligned}$$

Analogously, we get $\|f|L_{p^{\nu_j},q}(\Omega)\| \le M\|f|L_{p^{\tau_j},q}(\Omega)\|$.

Let $j_1 > j_0$ with $1 < r2^{j_1}$. Then, we have

$$\frac{1}{p^{\mu_j}} + \frac{1}{p^{\eta_{j_1+j}}} = \frac{1}{p^{\tau_{j_1+j}}} - \frac{1}{p^{\nu_j}} = \frac{r2^{j_1}-1}{r2^{j_1+j}} \to 0 \quad (j \to \infty).$$

So, similarly as above, we can prove

$$\|f|L_{p^{\mu_j},q}(\Omega)\| \le M_1\|f|L_{p^{\eta_{j_1+j}},q}(\Omega)\|$$

and

$$\|f|L_{p^{\tau_{j_1+j}},q}(\Omega)\| \le M_1\|f|L_{p^{\nu_j},q}(\Omega)\|.$$

Now, we apply Theorem 3.1.2 (in the case $\bar{\alpha} \leq 0$) and Theorem 3.2.2 (in the case $\bar{\alpha} \geq 0$) to the couple $\big(L_\infty(\Omega), L_r(\Omega)\big)$. This gives us a characterization with spaces $L_{p^{\eta_j},q}(\Omega)$ (in the case $\bar{\alpha} \leq 0$) and $L_{p^{\tau_j},q}(\Omega)$ (in the case $\bar{\alpha} \geq 0$). As in the proof of Corollary 3.4.1 the result follows with the help of the above estimates. □

Example 3.4.2. (i) Let $N = 1$ and $\bar{\alpha} = (\alpha)$. If $\alpha < 0$ we get

$$\|f|L_{p,q}(\log L)_\alpha(\Omega)\| \sim \left(\sum_{j=j_0}^{\infty} 2^{j\alpha q} \, \|f|L_{p^{\mu_j},q}(\Omega)\|^q\right)^{1/q}.$$

This is the result from [4, pp. 77].

(ii) Let $N = 2$ and $\bar{\alpha} = (0, \alpha)$ with $\alpha < 0$. Then it follows

$$\|f|L_{p,0,\alpha,q}(\Omega)\| \sim \left(\sum_{j=j_0}^{\infty} j^{\alpha q} \, \|f|L_{p^{\mu_j},q}(\Omega)\|^q\right)^{1/q}.$$

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Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.